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The Umbral Symbolic Method for Supersymmetric Tensors

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0. INTRODUCTION

In the preface to the first edition of *The Classical Groups* (1938), Hermann Weyl writes: "Important though the general concepts and propositions may be with which the modern industrious passion for axiomatizing and generalizing has presented us, in algebra perhaps more than anywhere else, nevertheless I am convinced that the special problems

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in all their complexity constitute the stock and the core of mathematics; and to master their difficulties requires on the whole the harder labor." While paying homage to 19th century mathematicians' bent for explicit computations and detailed analysis of specific phenomena, this passage in Weyl's book clearly anticipates the current revival of the constructive approach to algebra with its store of combinatorial methods and algorithmic techniques. Actually, it should be recognized that the revival of combinatorial methods has been guided and, in some sense, made possible by the conceptual clarity provided by modern abstract algebraic theories; on the other hand, it has to be granted that most of the main present-day achievements—such as Straightening Laws, Schensted–Schutzenberger–Knuth correspondences, and Buchberger's algorithm for Gröbner bases, to name but a few—have been inspired by the pioneering intuitions (often supported by rather obscure computations) of such mathematicians as Capelli, Cayley, Clebsch, Gordan, Grassmann, Macaulay, E. Pascal, Sylvester, and A. Young.

Among the main discoveries of 19th century invariant theory, the *symbolic method* was one of the most typical and fruitful; this method "in theory allowed the computation of all invariants by a quasi-mechanical process" (Dieudonné and Carrel [19]). However, in spite of the "widespread circulation of its applications" (Weyl [57, p. 247]), it has been marooned for several decades in a neglected corner. Like several other techniques of the pre-Hilbert invariant theory, the symbolic method was spoiled by a lack of structural algebraic understanding of the problems to be dealt with, and thus later brushed off as a mere technical device, suited for interesting but limited purposes. As Dieudonné and Carrel state, "... it was realized that, except in very few simple cases, the actual computation would lead to enormous labor, disproportionated to the interest of the outcome, especially in a period when all calculation were done by hand." Worse yet, the classical symbolic method works on symmetric tensors, but it does not apply to skew-symmetric tensors; Weitzenböck's lonely attempt to develop a true analog of the symbolic method for skew-symmetric tensors —the so-called "Komplex-Symbolik"—failed and only a few invariants (other than those previously known) were computed by such a technique (see, e.g., [23]).

To the best of our knowledge, Weyl was the first to recognize a neat connection between the Arhonold symbols (i.e., the classical symbolic method) and Weitzenböck's Komplex-Symbolik in his discussion of "a general method for including contravariant arguments" [57, p. 49]; he even attempted an extension of the technique to tensors belonging to arbitrary symmetry classes [57, p. 245]. Nonetheless, we submit that the problem of finding a unified and effective symbolic method for both symmetric and skew-symmetric tensors remained unsolved until the publication of

Grosshans, Rota, and Stein's book *Invariant Theory and Superalgebras* in 1987 [23]. The key idea of this work is, *a posteriori*, quite simple. Specifically, the algebra generated by the symbolic variables must be a *supersymmetric* algebra; that is, the symbolic letters have to be endowed with a $\mathbb{Z}_2$-gradation (or *signature*) which depends on the fact that the tensor they represent is either a symmetric or a skew-symmetric tensor. The commutation rules in the (supersymmetric) symbolic algebra depend on the signature of the variables. The algebra of polynomial functions on a finite set of "generic" symmetric and skew-symmetric tensors on a vector space V is viewed as the epimorphic image of the symbolic algebra under a $\mathbf{GL}(V)$-equivariant map $\mathbf{U}$, which is called the *symbolic operator*; thus, the problem of finding the joint invariants of these tensors is reduced to the problem of finding the invariants of the supersymmetric symbolic algebra. By using the *Straightening Formula for the supersymmetric letterplace algebra* [23], the symbolic invariants are shown to be polynomials in "supersymmetric brackets"; therefore, the First Fundamental Theorem remains valid for the joint invariants of a set of both symmetric and skew-symmetric tensors after the traditional brackets (i.e., determinants) are just replaced by their supersymmetric counterparts. The supersymmetric version of the straightening formula is the main technical innovation in Grosshans, Rota, and Stein's work. It is worth remarking that this general result can be given a simpler proof [10] than those given in [18, 20].

Thus, the ideas and the philosophy of "supersymmetric algebra" turn out to be effective when applied to ordinary invariant theory, and the question arises of whether the symbolic method can be further extended to the superalgebraic setting of Lie superalgebra actions on varieties of supersymmetric tensors [29, 30, 32, 34, 49].

In this work, we carry out a part of this program, by deriving the portion of the theory that applies to the actions of *general linear* Lie superalgebras. Limited as this approach may appear, we submit that it yields an attractive dividend even when restricted to the study of classical problems dealing with ordinary rather than supersymmetric variables. For instance, the technique of *virtual variables*—whose idea can be traced back to Capelli's work [13]—acquires a special suppleness when the virtual variables are allowed to have a different signature than the signature of the variables one starts with. This device often radically cuts down the amount of computation [8–12]. Furthermore, the superalgebraic method permits us to establish natural correspondences that were formely missing; a typical example is found in De Concini and Procesi's straightening formulas for Pfaffians and for Gramians [17]. The striking resemblance between these formulas has been noted by several authors [1, 2, 7, 15]; from the superalgebraic point of view, these formulas turn out to be "symbolic images" of one and the same straightening formula for the minors of a generic matrix [18, 21], under the involutorial operation of "flipping signatures" of symbolic vectors.

In principle, supersymmetric methods can be put to use whenever symmetric algebra or exterior algebra methods are called for. Since most of the cumbersome oddities arising from the separation between symmetry and skew-symmetry fade away, the superalgebraic approach brings the symbolic method closer to its computational sources and, therefore, we hope it will contribute to a deeper understanding of the combinatorial core of the theory.

The paper is organized as follows.

In the first section, we introduce the objects of our study, *the basic plethystic superalgebras* $\mathbf{S}(\mathbf{S}^k(V))$ and $\Lambda(\mathbf{S}^k(V))$. These superalgebras are the result of the compositions of the "supersymmetric" and the "super-exterior" functors $\mathbf{S}$ and Λ; even though these functors are related by the rather trivial operation of "flipping signatures" on the underlying $\mathbb{Z}_2$-graded vector space V, it has to be stressed that their compositions give rise to radically different modules under the action of the general linear Lie superalgebra $\mathrm{pl}(V)$. It is of particular interest to note that the classical plethystic algebra $\mathrm{Sym}(\Lambda^k(V))$ appears as a special case of either $\mathbf{S}(\mathbf{S}^k(V))$ or $\Lambda(\mathbf{S}^k(V))$ depending on k being even or odd, and so does $\Lambda(\Lambda^k(V))$. We call these algebras the basic plethystic algebras since, in the classical case, the characters of the $\mathbf{GL}(V)$-representations $\mathrm{Sym}^n(\mathrm{Sym}^k(V))$, $\mathrm{Sym}^n(\Lambda^k(V))$, $\Lambda^n(\mathrm{Sym}^k(V))$, and $\Lambda^n(\Lambda^k(V))$ are plethystic compositions of elementary and complete homogeneous symmetric functions. Thus, the problem of describing the structure of these representations as semisimple $\mathbf{GL}(V)$-modules is equivalent to the problem of expanding plethystic compositions of these symmetric functions into linear combinations of Schur functions [38–40].

In the classical theory, the symbolic method is ultimately a multilinearization process; the idea is that a generic symmetric tensor of step n over $\mathrm{Sym}^k(V)$ can be treated as a special symmetric product of vectors, by taking *n distinct copies* of the underlying vector space V. Owing to the natural isomorphism $V^{\oplus^n} \cong W \otimes V$, $\dim(W) = n$, the representation theory of $\mathrm{Sym}^n(\mathrm{Sym}^k(V))$ can be inferred from the simpler theory that holds for the so-called *letterplace algebra* $\mathrm{Sym}(W \otimes V)$ [21, 23]. In 19th century terminology, W is the space spanned by the *symbolic letters* and the elements of $W \otimes V$ are called the *Arhonold symbols* [13, 19, 53].

The extension of this technique to the superalgebraic setting—which is described in Section 2—is delicate. Not unexpectedly, the fact that the vector space V is allowed to be a $\mathbb{Z}_2$-graded vector space forces us to consider a symbolic letterplace superalgebra whose "places" are also $\mathbb{Z}_2$-graded, and the action of the general linear Lie superalgebra $\mathrm{pl}(V)$ must be implemented by *superpolarizations* [8]. Furthermore, *two different* symbolic operators $\mathbf{U}$ and $\mathbf{U}'$ must be defined depending on the superalgebra—either $\mathbf{S}(\mathbf{S}^k(V))$ or $\Lambda(\mathbf{S}^k(V))$—under investigation. Though the operators $\mathbf{U}$ and $\mathbf{U}'$ are quite different in their behaviour, they admit parallel definitions once the trick (a new trick, we believe) of introducing a $\mathbb{Z}_2$-graded *mock place* is used. Mock places give rise to auxiliary letterplace variables that do not change the representation theory of the symbolic algebra. Their use is not necessary in the classical setting; however, it is essential to make $\mathbf{U}$ and $\mathbf{U}'$ equivariant maps when the action of a Lie superalgebra is taken into consideration. The use of the symbolic operators $\mathbf{U}$ and $\mathbf{U}'$ proves to be effective. In fact, a complete decomposition of the symbolic letterplace superalgebra into $\mathrm{pl}(V)$-irreducibles is known; the concrete way to describe such a decomposition is provided by the *Gordan–Capelli series* [8, 9], which is the basis of our work. The Gordan–Capelli series is the characteristic-zero counterpart of the Straightening Formula; it yields an *ordered* set of $\mathrm{pl}(V)$-irreducibles which decompose the natural filtration associated with the Straightening Formula [8, 9].

When working in a characteristic-zero setting, one is faced with the amazing fact that two essentially different Gordan–Capelli series for the same letterplace superalgebra are available. For example, one can decompose the algebra $\mathrm{Sym}(W \otimes V)$ either by using modules spanned by "symmetrized bideterminants" or by using modules spanned by "skew-symmetrized bipermanents." This phenomenon is not unexpected; it is an instance of the duality between Schur modules and Weyl modules [3, 4, 11] and the connection with the classical main involution of the ring of symmetric functions [40] makes it even clearer. From a superalgebraic point of view, we exploit this duality by introducing the *$\mathbb{Z}_2$-companion symbolic operators* $\bar{\mathbf{U}}$ and $\bar{\mathbf{U}}'$. The definitions of these operators are formally the same as that of $\mathbf{U}$ and $\mathbf{U}'$, except that the signatures of both symbolic letters and places must be flipped in the letterplace superalgebra. Even though this construction may appear irrelevant, it entails some notable consequences, especially in the theory of superalgebras on *supersymmetric matrices* (i.e., elements of $\mathbf{S}^2(V)$).

The study of the plethystic superalgebras $\mathbf{S}(\mathbf{S}^2(V))$ and $\Lambda(\mathbf{S}^2(V))$ is the theme of Sections 3 and 4. The special cases $\mathrm{Sym}(\mathrm{Sym}^2(V_0))$ and $\mathrm{Sym}(\Lambda^2(V_1))$ have been widely investigated by several authors, both in the characteristic-zero and in the characteristic-free settings [1–3, 7, 15, 17, 33, 37]. De Concini and Procesi's straightening formulas for Pfaffians and for inner products play a crucial role; recently, these results have been extended by Rota and Stein to the (characteristic-free) superalgebraic settings of "supersymplectic algebras" [47] and of "supersymmetric Hilbert spaces" [48], respectively. We prove these formulas to be the "images," either under the action of the symbolic operator $\mathbf{U}$ or under the action of its $\mathbb{Z}_2$-companion $\bar{\mathbf{U}}$, of the same straightening formula for the supersymmetric letterplace superalgebra [23, 8, 9]. In particular, it follows that the basic identities and the canonical forms that hold for polynomials involving Pfaffians, inner products, and their "superanalogs" can be mechanically derived from the traditional Plücker relations for the minors of a generic matrix.

The choice between the operator $\mathbf{U}$ or its $\mathbb{Z}_2$-companion $\bar{\mathbf{U}}$ depends on the problem at hand; as a typical example, in Section 5 we derive in a few lines proofs of E. Pascal-type "linear" versions of the Second Fundamental Theorem for symplectic and orthogonal invariants.

The symbolic method yields a concrete description of all the irreducible $\mathrm{pl}(V)$-submodules of the basic plethystic superalgebras $\mathbf{S}(\mathbf{S}^k(V))$ and $\Lambda(\mathbf{S}^k(V))$; this result (Theorem 10) is a generalization of the First Fundamental Theorem of classical invariant theory (Section 8). The general problem of finding complete decompositions of $\mathbf{S}(\mathbf{S}^k(V))$ and $\Lambda(\mathbf{S}^k(V))$ is still open for arbitrary values of k. A partial solution of this problem is provided by the Second Fundamental Theorem, which describes the kernels of the symbolic operators, and relates the *length* of the isotypic components of the basic plethystic superalgebras to the linear dimensions of the spaces of the invariants of the modules $\mathrm{Specht}_k^{\pm}(\lambda)$, $\lambda \vdash nk$, with respect to the action of the symmetric group on the symbolic letters (Section 9). We call these modules the *generalized Specht modules* since, in the case $k=1$, they yield the ordinary irreducible representations of the symmetric group. This result is unsatisfactory from a computational point of view. The modules $\mathrm{Specht}_k^{\pm}(\lambda)$ are representations of the *wreath-products* $S_n \sim S_k$ in disguise [28, 40] and no general formula to compute the dimensions of the spaces of their invariants is known; we believe that the description of the structure of the generalized Specht modules could be the last step towards a combinatorial understanding of the symbolic method. The core of the problem can be better understood by a closer inspection of the case of supersymmetric matrices, that is $k=2$. The superalgebras $\mathbf{S}(\mathbf{S}^2(V))$ and $\Lambda(\mathbf{S}^2(V))$ can be effectively decomposed by using the technique of *regularization algorithms* which is introduced in this work. Roughly speaking, a regularization algorithm transforms a symbolic presentation into an equivalent one with prescribed properties. In the special case of supersymmetric matrices, we establish regularization algorithms that are indeed *normal form* algorithms for the quotient of the letterplace superalgebra modulo the kernel of the symbolic operator. Specifically, using the fact that $\mathbf{S}(\mathbf{S}^2(V))$ and $\Lambda(\mathbf{S}^2(V))$ are multiplicity-free $\mathrm{pl}(V)$-modules, one can "regularize" the Gordan–Capelli series of the letterplace superalgebra by choosing just one letter-tableau for any Young shape (Propositions 8 and 20).

The *umbral calculus* proper is discussed in Section 7. In Silvester's language, "umbrae" were what we now call linear functionals; at bottom, the umbral calculus is the symbolic method working on the dual space. It is founded upon the amazing phenomenon that "indices become exponents" [13, 19, 22, 23, 36, 44, 51, 56]. When this idea is extended to the superalgebraic setting, the problem is that the notion of a contragradient action of a Lie superalgebra is different from the ordinary one (see, e.g., Scheunert [49]) and, in general, it involves complicated sign computations. We avoid this difficulty by introducing an *even* pairing that allows the contragradient action to be implemented by (right) superderivations (Theorem 7); thus, the *same* theory that holds for the symbolic method is still valid for the umbral calculus, modulo the supertransposition automorphism of the general linear Lie superalgebra (Proposition 25 and Theorem 8).

In the final section of the paper, we outline the extension of the theory to plethystic algebras on direct sums of spaces of both covariant and contravariant symmetric and skew-symmetric tensors. In classical language, the invariants of these algebras are examples of *concomitants*. As expected, the problem reduces to the study of symbolic letterplace superalgebras whose places are partitioned into a set of covariant and a set of contravariant ones. By exploiting a supersymmetric version of Weyl's "general method for including contravariant arguments" [57, p. 49], we derive a regularization algorithm (Proposition 35) which yields a generalization of the First Fundamental Theorem [23] for the covariants of a set of both symmetric and skew-symmetric tensors (Theorem 13).

We have tried to give a self-contained presentation. However, for reasons of space, we could not summarize the preliminary representation theory of letterplace superalgebras. For the basic facts of this theory, such as Straightening Formulas and Gordan–Capelli series, and for the definitions of biproducts, bitableaux, and symmetrized bitableaux, we refer to [23, 8, 9]. An elementary use of Capelli's technique of "virtual variables" is implicit through the paper, whenever the number of negative places is sufficiently large; for a rigorous foundation of this method, we refer to [10].

1. The Basic Plethystic Algebras

Let $V = V_0 \oplus V_1$ be a $\mathbb{Z}_2$-graded finite dimensional vector space with $\dim(V) = d$ over a field $\mathbb{K}$ of characteristic zero. Elements in $V_0 \cup V_1$ are $\mathbb{Z}_2$-homogeneous; we will not repeat the word $\mathbb{Z}_2$-homogeneous in obvious situations. If $v \in V_i$, then we write $|v|$ for the $\mathbb{Z}_2$-gradation (or *signature*) of the vector v. We always consider $\mathbb{K}$ as a $\mathbb{Z}_2$-graded vector space such that $\mathbb{K} = \mathbb{K}_0$. For convenience, a vector v in V_0 (or V_1) is sometimes denoted as v^+ (or v^-). The *general linear Lie superalgebra of* V is the pair

$\mathrm{pl}(V) = (\mathrm{End}_{\mathbb{K}}(V), [\ ,\ ])$, where $\mathrm{End}_{\mathbb{K}}(V) = (\mathrm{End}_{\mathbb{K}}(V))_0 \oplus (\mathrm{End}_{\mathbb{K}}(V))_1$, and $(\mathrm{End}_{\mathbb{K}}(V))_j$ is the vector space spanned by the set of all linear endomorphisms φ of V such that $\varphi(V_i) \subseteq V_{i+j}$, $i, j \in \mathbb{Z}_2$. Such a φ will be called a *$\mathbb{Z}_2$-homogeneous linear transformation* of degree $|\varphi| = j$. The bracket $[\ ,\ ]$ is defined as

$$[\varphi, \psi] = \varphi \circ \psi - (-1)^{|\varphi|\,|\psi|}\, \psi \circ \varphi$$

for every pair of $\mathbb{Z}_2$-homogeneous linear transformations φ and ψ. Next we define the involution "$^{-}$" by flipping the signs of vectors in V, i.e., setting $\bar{V} = \bar{V}_0 \oplus \bar{V}_1$, where $\bar{V}_0 = V_1$ and $\bar{V}_1 = V_0$. The vector space $\bar{V}$ is called the *$\mathbb{Z}_2$-graded companion of V*. Given a vector $v \in V$, we denote by the bold face $\boldsymbol{v}$ the corresponding vector in $\bar{V}$. This simple definition of $\bar{V}$ enables us to get naturally the standard involution for symmetric functions. Clearly, we have

$$\bar{\bar{V}} = V \qquad \text{and} \qquad \mathrm{pl}(V) = \mathrm{pl}(\bar{V}).$$

Both V and $\bar{V}$ are $\mathrm{pl}(V)$-modules. Note that they are the same vector space and the actions of $\mathrm{pl}(V)$ on V and on $\bar{V}$ are also the same; therefore the two modules can be regarded as the same.

The *supersymmetric algebra* of V is defined to be

$$\mathbf{S}(V) = \mathrm{Sym}(V_0) \otimes \Lambda(V_1);$$

this is indeed a $\mathbb{Z}_2$-commutative and associative superalgebra if we take the grading

$$\mathbf{S}(V)_0 = \mathrm{Sym}(V_0) \otimes \left(\bigoplus_{k \in \mathbb{N}} \Lambda^{2k}(V_1) \right),$$

$$\mathbf{S}(V)_1 = \mathrm{Sym}(V_0) \otimes \left(\bigoplus_{k \in \mathbb{N}} \Lambda^{2k+1}(V_1) \right),$$

and let the product be $u \otimes v \cdot u' \otimes v' = uu' \otimes vv'$ for u, $u' \in \mathrm{Sym}(V_0)$ and $v, v' \in \Lambda(V_1)$. One can check that the relation

$$\omega \cdot \omega' = (-1)^{|\omega|\,|\omega'|}\, \omega' \cdot \omega$$

holds for each pair of $\mathbb{Z}_2$-homogeneous elements ω, $\omega' \in \mathbf{S}(V)$, where $|\omega| = i$ if $\omega \in \mathbf{S}(V)_i$, $i \in \mathbb{Z}_2$. The supersymmetric algebra $\mathbf{S}(V)$ also endows a $\mathbb{Z}$-grading $\mathbf{S}(V) = \bigoplus_{n \in \mathbb{N}} \mathbf{S}^n(V)$, where

$$\mathbf{S}^n(V) = \bigoplus_h (\mathrm{Sym}^h(V_0) \otimes \Lambda^{n-h}(V_1)).$$

The $\mathbb{Z}$-grading and $\mathbb{Z}_2$-grading of $\mathbf{S}(V)$ are compatible in the sense that $\mathbf{S}^n(V) = \mathbf{S}^n(V)_0 \oplus \mathbf{S}^n(V)_1$ and $\mathbf{S}(V)_i = \bigoplus_{n \in \mathbb{N}} \mathbf{S}^n(V)_i$, $i \in \mathbb{Z}_2$.

PROPOSITION 1. $\mathbf{S}(V)$ *is a* $\mathrm{pl}(V)$*-module.*

Proof. Let us define an even representation $\rho: \mathrm{pl}(V) \to \mathrm{pl}(\mathbf{S}(V))$ as follows. Given a $\mathbb{Z}_2$-homogeneous $\varphi \in \mathrm{pl}(V)$, the action of $\rho(\varphi)$ is the unique right superderivation of $\mathbf{S}(V)$

$$\rho(\varphi)(\omega \cdot \omega') = (-1)^{|\varphi|\,|\omega'|}\, \rho(\varphi)\, \omega \cdot \omega' + \omega \cdot \rho(\varphi)\, \omega'$$

such that $\rho(\varphi)(v) = \varphi(v)$ for $v \in V$. To see the relation $\rho([\varphi, \psi]) = [\rho(\varphi), \rho(\psi)]$, it is sufficient to check that $[\rho(\varphi), \rho(\psi)]$ is still a right superderivation of signature $(-1)^{|\varphi| + |\psi|}$. First, we have

$$\begin{aligned}(\rho(\varphi) \circ \rho(\psi))(\omega \cdot \omega') &= \rho(\varphi)((-1)^{|\psi|\,|\omega'|}\, \rho(\psi)\, \omega \cdot \omega' + \omega \cdot \rho(\psi)\, \omega') \\ &= (-1)^{|\psi|\,|\omega'| + |\varphi|\,|\omega'|}\, (\rho(\varphi) \circ \rho(\psi))\, \omega \cdot \omega' \\ &\quad + (-1)^{|\psi|\,|\omega'|}\, \rho(\psi)\, \omega \cdot \rho(\varphi)\, \omega' \\ &\quad + (-1)^{|\varphi|\,(|\psi| + |\omega'|)}\, \rho(\varphi)\, \omega \cdot \rho(\psi)\, \omega' \\ &\quad + \omega \cdot (\rho(\varphi) \circ \rho(\psi))\, \omega'.\end{aligned}$$

Therefore,

$$\begin{aligned}&[\rho(\varphi), \rho(\psi)]\,(\omega \cdot \omega') \\ &\quad = (\rho(\varphi) \circ \rho(\psi))(\omega \cdot \omega') - (-1)^{|\varphi|\,|\psi|}(\rho(\psi) \circ \rho(\varphi))(\omega \cdot \omega') \\ &\quad = (-1)^{(|\varphi| + |\psi|)\,|\omega'|}\, [\rho(\varphi), \rho(\psi)]\, \omega \cdot \omega' + \omega \cdot [\rho(\varphi), \rho(\psi)]\, \omega',\end{aligned}$$

which indicates that $[\rho(\varphi), \rho(\psi)]$ is the right superderivation of the desired signature. ∎

One can also use left superderivations to define the $\mathrm{pl}(V)$-module $\mathbf{S}(V)$, which is $\mathrm{pl}(V)$-isomorphic to the one defined by right superderivations (see the Appendix for more details).

Next we define the *superexterior algebra* $\Lambda(V)$ to be $\mathbf{S}(\bar{V})$. We remark that $\overline{\mathbf{S}(V)}$ cannot be identified with $\mathbf{S}(\bar{V})$; for example, the element $v_1^+ v_2^+$ belongs to $\overline{\mathbf{S}(V)}_1$ while the element $v_1^- v_2^-$ belongs to $\mathbf{S}(\bar{V})_0$. Although V and $\bar{V}$ are the same $\mathrm{pl}(V)$-module, $\mathbf{S}(V)$ and $\mathbf{S}(\bar{V})$ are different $\mathrm{pl}(V)$-modules. To see this, let $e_1, e_2, ..., e_d$ be a $\mathbb{Z}_2$*-homogeneous basis* of V; that is, either $e_i \in V_0$ or $e_i \in V_1$. Denote by E_{ij} the linear transformation such that $E_{ij} e_k = \delta_{jk} e_i$. Obviously $|E_{ij}| = |e_i| + |e_j|$. Then we have $\rho(E_{n^+1^-})\, e_1^- e_2^+ = e_n^+ e_2^+$ in $\mathbf{S}(V)$, while $\rho(E_{n^-1^+})\, e_1^+ e_2^- = -e_n^- e_2^-$ in $\mathbf{S}(\bar{V})$. A final remark is that the vector space spanned by the right superderivations of $\mathbf{S}(V)$ is a Lie subsuperalgebra of $\mathrm{pl}(\mathbf{S}(V))$.

Iterating the operators $\mathbf{S}$ and Λ, we obtain the four *basic plethystic algebras*:

(I) $\mathbf{S}(\mathbf{S}(V))$,
(II) $\Lambda(\mathbf{S}(V)) = \mathbf{S}(\overline{\mathbf{S}(V)})$,
(III) $\mathbf{S}(\Lambda(V)) = \mathbf{S}(\mathbf{S}(\bar{V}))$,
(IV) $\Lambda(\Lambda(V)) = \mathbf{S}(\overline{\mathbf{S}(\bar{V})})$.

Substituting V by $\bar{V}$, it can be seen that there are only two essentially different types among the four: type (I) and type (II). To get the *classical basic plethystic algebras*, let us expand (I):

$$\begin{aligned}\mathbf{S}^n(\mathbf{S}^k(V)) &= \bigoplus_h [\mathrm{Sym}^h(\mathbf{S}^k(V)_0) \otimes \Lambda^{n-h}(\mathbf{S}^k(V)_1)] \\ &= \bigoplus_h \Bigg[\mathrm{Sym}^h \Bigg(\bigoplus_m (\mathrm{Sym}^{k-2m}(V_0) \otimes \Lambda^{2m}(V_1)) \Bigg) \\ &\quad \otimes \Lambda^{n-h} \Bigg(\bigoplus_m (\mathrm{Sym}^{k-2m-1}(V_0) \otimes \Lambda^{2m+1}(V_1)) \Bigg) \Bigg]\end{aligned}$$

When $V_1 = \mathbf{0}$, since $\Lambda^0(\mathbf{0}) = \mathbb{K}$ and $\Lambda^k(\mathbf{0}) = \mathbf{0}$ for $k > 0$, we get

$$\begin{aligned}\Lambda^{n-h} \Bigg(\bigoplus_m (\mathrm{Sym}^{k-2m-1}(V_0) \otimes \Lambda^{2m+1}(V_1)) \Bigg) &= \Lambda^{n-h}(\mathbf{0}) \\ &= \begin{cases} \mathbb{K} & \text{if } h = n, \\ \mathbf{0} & \text{if } h < n; \end{cases}\end{aligned}$$

$$\mathrm{Sym}^n \Bigg(\bigoplus_m (\mathrm{Sym}^{k-2m}(V_0) \otimes \Lambda^{2m}(V_1)) \Bigg) = \mathrm{Sym}^n(\mathrm{Sym}^k(V_0));$$

therefore

$$\mathbf{S}^n(\mathbf{S}^k(V_0)) = \mathrm{Sym}^n(\mathrm{Sym}^k(V_0)). \tag{1}$$

When $V_0 = \mathbf{0}$, since $\mathrm{Sym}^0(\mathbf{0}) = \mathbb{K}$ and $\mathrm{Sym}^k(\mathbf{0}) = \mathbf{0}$ for $k > 0$, we get by a similar argument

$$\mathbf{S}^n(\mathbf{S}^k(V_1)) = \begin{cases} \mathrm{Sym}^n(\Lambda^k(V_1)) & \text{if } k \text{ is } \textit{even}, \\ \Lambda^n(\Lambda^k(V_1)) & \text{if } k \text{ is } \textit{odd}. \end{cases} \tag{2}$$

Similarly

$$\Lambda^n(\mathbf{S}^k(V_0)) = \Lambda^n(\mathrm{Sym}^k(V_0)), \tag{3}$$

and

$$\Lambda^n(\mathbf{S}^k(V_1)) = \begin{cases} \Lambda^n(\Lambda^k(V_1)) & \text{if } k \text{ is } \textit{even}, \\ \mathrm{Sym}^n(\Lambda^k(V_1)) & \text{if } k \text{ is } \textit{odd}. \end{cases} \tag{4}$$

It is of particular interest to note that the classical basic plethystic algebra $\mathrm{Sym}^n(\Lambda^k(V_1))$ appears either in (2) or in (4) depending on k being *even* or *odd*; so does $\Lambda^n(\Lambda^k(V_1))$.

The plethystic algebras $\mathbf{S}(\mathbf{S}(V))$ and $\Lambda(\mathbf{S}(V))$ can be given combinatorial presentations as follows. Let $\mathscr{P}=\{e_1, e_2, ..., e_d\}$ be a $\mathbb{Z}_2$-homogeneous basis of V. Denote by $\mathrm{Mon}(\mathscr{P})$ the free monoid generated by $\mathscr{P}$. Define the $\mathbb{Z}_2$-grade $|\omega|$ of an element $\omega \in \mathrm{Mon}(\mathscr{P})$ recursively by $|\omega\omega'| = |\omega| + |\omega'|$. Consider the semigroup algebra $\mathbb{K}[\mathrm{Mon}(\mathscr{P})]$ of $\mathrm{Mon}(\mathscr{P})$. It is an associative superalgebra; in fact, $\mathbb{K}[\mathrm{Mon}(\mathscr{P})] = T(V)$, where $T(V)$ is the tensor algebra of V. Let J be the ideal of $\mathbb{K}[\mathrm{Mon}(\mathscr{P})]$ generated by the elements

$$\omega\omega' - (-1)^{|\omega|\,|\omega'|}\,\omega'\omega, \qquad \omega, \omega' \in \mathrm{Mon}(\mathscr{P}).$$

PROPOSITION 2. $\mathbb{K}[\mathrm{Mon}(\mathscr{P})]/J \cong \mathbf{S}(V)$.

Next we describe $\mathbf{S}(\mathbf{S}^k(V))$. Set

$$[\mathscr{P}]_k = \{[\omega];\, \omega \in \mathrm{Mon}(\mathscr{P}),\, \mathrm{length}(\omega) = k\}.$$

Define $|[\omega]| = |\omega|$; then $\mathrm{Mon}([\mathscr{P}]_k)$ is also a $\mathbb{Z}_2$-graded monoid and hence $\mathbb{K}[\mathrm{Mon}([\mathscr{P}]_k)]$ is again an associative superalgebra. Let I be the ideal of $\mathbb{K}[\mathrm{Mon}([\mathscr{P}]_k)]$ generated by elements of the types

(i) $[\omega][\omega'] - (-1)^{|\omega|\,|\omega'|}\,[\omega'][\omega]$, $[\omega], [\omega'] \in [\mathscr{P}]_k$,

(ii) $[e_{i_1} \cdots e_{i_h} e_{i_{h+1}} \cdots e_{i_k}] = (-1)^{|e_{i_h}|\,|e_{i_{h+1}}|}\,[e_{i_1} \cdots e_{i_{h+1}} e_{i_h} \cdots e_{i_k}]$.

PROPOSITION 3. $\mathbb{K}[\mathrm{Mon}([\mathscr{P}]_k)]/I \cong \mathbf{S}(\mathbf{S}^k(V))$.

To describe $\Lambda(\mathbf{S}^k(V)) = \mathbf{S}(\overline{\mathbf{S}^k(V)})$, let $\bar{I}$ be the ideal of $\mathbb{K}[\mathrm{Mon}([\mathscr{P}]_k)]$ generated by elements of the types

(i) $[\omega][\omega'] - (-1)^{(|\omega|+1)(|\omega'|+1)}\,[\omega'][\omega]$, $[\omega], [\omega'] \in [\mathscr{P}]_k$,

(ii) $[e_{i_1} \cdots e_{i_h} e_{i_{h+1}} \cdots e_{i_k}] = (-1)^{|e_{i_h}|\,|e_{i_{h+1}}|}\,[e_{i_1} \cdots e_{i_{h+1}} e_{i_h} \cdots e_{i_k}]$.

PROPOSITION 4. $\mathbb{K}[\mathrm{Mon}([\mathscr{P}]_k)]/\bar{I} \cong \Lambda(\mathbf{S}^k(V))$.

2. THE SYMBOLIC METHOD FOR $\mathbf{S}^n(\mathbf{S}^k(V))$ AND $\Lambda^n(\mathbf{S}^k(V))$

Let $\mathscr{L} = \{\alpha_1^+, \alpha_2^+, ..., \alpha_n^+\}$ be a set; let the (positive) elements α_i be called *letters*. Let us call the elements of the set $\mathscr{P} = \{e_1, e_2, ..., e_d\}$ *places*. Add a *mock place* x to $\mathscr{P}$ and set $\mathscr{P}_x = \mathscr{P} \dot\cup \{x\}$. The *letterplace superalgebra*

$\mathrm{Super}[\mathscr{L} \mid \mathscr{P}_x]$ is defined as in [23]. Denote by $\mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}_x]$ the subspace of $\mathrm{Super}[\mathscr{L} \mid \mathscr{P}_x]$ spanned by the monomials

$$\begin{aligned}(\alpha_1^{(k+1)} \mid xe_{i_1} \cdots e_{i_k})(\alpha_2^{(k+1)} \mid xe_{j_1} \cdots e_{j_k}) \cdots (\alpha_n^{(k+1)} \mid xe_{h_1} \cdots e_{h_k})\\ = (\alpha_1 \mid x)(\alpha_1 \mid e_{i_1}) \cdots (\alpha_1 \mid e_{i_k})(\alpha_2 \mid x)(\alpha_2 \mid e_{j_1}) \cdots (\alpha_2 \mid e_{j_k}) \cdots\\ \cdots (\alpha_n \mid x)(\alpha_n \mid e_{h_1}) \cdots (\alpha_n \mid e_{h_k}),\end{aligned}$$

where $\alpha_i^{(k+1)}$ is the $(k+1)$th divided power, i.e., $\alpha_i^{(k+1)} = \alpha_i^{k+1}/(k+1)!$. Define the *symbolic linear operators*

$$\mathbf{U}: \mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}_{x^+}] \to \mathbf{S}^n(\mathbf{S}^k(V))$$

and

$$\mathbf{U}': \mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}_{x^-}] \to \Lambda^n(\mathbf{S}^k(V))$$

to be the ones linearly extended from

$$\begin{aligned}\mathbf{U}((\alpha_1^{(k+1)} \mid x^+e_{i_1} \cdots e_{i_k}) \cdots (\alpha_n^{(k+1)} \mid x^+e_{h_1} \cdots e_{h_k}))\\ = [e_{i_1} \cdots e_{i_k}] \cdots [e_{h_1} \cdots e_{h_k}]\end{aligned} \tag{5}$$

and

$$\begin{aligned}\mathbf{U}'((\alpha_1^{(k+1)} \mid x^-e_{i_1} \cdots e_{i_k}) \cdots (\alpha_n^{(k+1)} \mid x^-e_{h_1} \cdots e_{h_k}))\\ = [e_{i_1} \cdots e_{i_k}] \cdots [e_{h_1} \cdots e_{h_k}]\end{aligned} \tag{6}$$

PROPOSITION 5. *The symbolic operators* $\mathbf{U}$ *and* $\mathbf{U}'$ *are well defined and surjective.*

Proof. The key observation is that the signature of $(\alpha_i^{(k+1)} \mid x^+e_{j_1} \cdots e_{j_k})$ in $\mathrm{Super}[\mathscr{L} \mid \mathscr{P}_x]$ is the same as the signature of $[e_{j_1} \cdots e_{j_k}]$ in $\mathbf{S}(\mathbf{S}(V))$; the signature of $(\alpha_i^{(k+1)} \mid x^-e_{j_1} \cdots e_{j_k})$ is the same as the signature of $[e_{j_1} \cdots e_{j_k}]$ in $\Lambda(\mathbf{S}(V))$, which is $|e_{j_1} \cdots e_{j_k}| + 1$. ∎

In fact, one can omit the mock place x^+ and just define

$$\mathbf{U}: \mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}] \to \mathbf{S}^n(\mathbf{S}^k(V)),$$

where $\mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}]$ is the $\mathbb{K}$-span of the monomials $(\alpha_1^{(k)} \mid e_{i_1} \cdots e_{i_k})(\alpha_2^{(k)} \mid e_{j_1} \cdots e_{j_k}) \cdots (\alpha_n^{(k)} \mid e_{h_1} \cdots e_{h_k})$, in a similar way without affecting anything. However, the mock place x^- is crucial in defining $\mathbf{U}'$, because the signatures of $[e_{j_1} \cdots e_{j_k}]$ in $\Lambda(\mathbf{S}(V))$ are flipped with respect to the signature of $(\alpha_j^{(k)} \mid e_{j_1} \cdots e_{j_k})$ in $\mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}]$.

The letterplace algebra $\mathrm{Super}[\mathscr{L} \mid \mathscr{P}_x]$ is also a $\mathrm{pl}(V)$-module and so is $\mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}_x]$, where the action is given by

$$E_{ij} \cdot \mathbf{m} = \mathbf{m}\ {}_{e_j e_i}\mathcal{D} \tag{7}$$

for $\mathbf{m} \in \mathrm{Super}[\mathscr{L} \mid \mathscr{P}_x]$, where ${}_{e_j e_i}\mathcal{D}$ is the right superderivation replacing e_j by e_i with $|{}_{e_j e_i}\mathcal{D}| = |e_j| + |e_i|$; see [9]. We make no distinction between E_{ij} and ${}_{e_j e_i}\mathcal{D}$ later on. On the other hand, both $\mathbf{S}^n(\mathbf{S}^k(V))$ and $\Lambda^n(\mathbf{S}^k(V))$ are also $\mathrm{pl}(V)$-modules, because $\mathbf{S}(V)$ is a $\mathrm{pl}(V)$-module via right superderivations. To make the actions of $\mathrm{pl}(V)$ on $\mathbf{S}^n(\mathbf{S}^k(V))$ and on $\Lambda^n(\mathbf{S}^k(V))$ more explicit, they are given by

$$\varphi \cdot ([\omega_1][\omega_2] \cdots [\omega_n])$$
$$= \sum_i (-1)^{|\varphi| (|\omega_{i+1}| + \cdots + |\omega_n|)} [\omega_1] \cdots [\varphi \cdot \omega_i] \cdots [\omega_n]$$

in $\mathbf{S}^n(\mathbf{S}^k(V))$ and

$$\varphi \cdot ([\omega_1][\omega_2] \cdots [\omega_n])$$
$$= \sum_i (-1)^{|\varphi| (|\omega_{i+1}| + \cdots + |\omega_n| + n - i)} [\omega_1] \cdots [\varphi \cdot \omega_i] \cdots [\omega_n]$$

in $\Lambda^n(\mathbf{S}^k(V))$, where φ is a $\mathbb{Z}_2$-homogeneous linear transformation of V, $[\omega_i] \in [\mathscr{P}]_k$, and the action $\varphi \cdot \omega_i$ takes place in the $\mathrm{pl}(V)$-module $\mathbf{S}^k(V)$.

THEOREM 1. *Both* $\mathbf{U}$ *and* $\mathbf{U}'$ *are* $\mathrm{pl}(V)$*-equivariant surjective linear operators; i.e.,* $\varphi \cdot \mathbf{U}(\mathbf{m}) = \mathbf{U}(\varphi \cdot \mathbf{m})$ *and* $\varphi \cdot \mathbf{U}'(\mathbf{m}) = \mathbf{U}'(\varphi \cdot \mathbf{m})$, *for every* $\varphi \in \mathrm{pl}(V)$.

Proof. The assertion follows immediately from the definitions, since the actions of $\mathrm{pl}(V)$ on $\mathrm{Super}[\mathscr{L} \mid \mathscr{P}_x]$, $\mathbf{S}(\mathbf{S}(V))$ and $\Lambda(\mathbf{S}(V))$ are all right superderivations and all the signs check. Note that the use of the mock place x^- is crucial in making $\mathbf{U}'$ $\mathrm{pl}(V)$-equivariant. ∎

Is it necessary to stick with positive symbols α_i^+ in the symbolic method? The answer is no. We can also define the $\mathbb{Z}_2$-companion symbolic operators $\bar{\mathbf{U}}$ and $\bar{\mathbf{U}}'$ using negative symbols as follows.

Let $\bar{\mathscr{L}} = \{\boldsymbol{\alpha}_1^-, \boldsymbol{\alpha}_2^-, ..., \boldsymbol{\alpha}_n^-\}$ and $\bar{\mathscr{P}}_{\mathbf{x}} = \{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_d, \mathbf{x}\}$, where the signature of $\mathbf{e}_i$ is the opposite to the one of e_i in $\mathscr{P}$, for every i. We also use boldface letters for tableaux on $\bar{\mathscr{L}}$ and $\bar{\mathscr{P}}$. As an associative superalgebra, $\mathrm{Super}[\mathscr{L} \mid \mathscr{P}_{x^-}]$ is the same as $\mathrm{Super}[\bar{\mathscr{L}} \mid \bar{\mathscr{P}}_{\mathbf{x}^+}]$, since each $(\alpha_i \mid e_j)$ and $(\alpha_i \mid x)$ keep the same signatures as $(\boldsymbol{\alpha}_i \mid \mathbf{e}_j)$ and $(\boldsymbol{\alpha}_i \mid \mathbf{x})$, respectively. Similarly, $\mathrm{Super}[\mathscr{L} \mid \mathscr{P}_{x^+}]$ is the same superalgebra as $\mathrm{Super}[\bar{\mathscr{L}} \mid \bar{\mathscr{P}}_{\mathbf{x}^-}]$. Denote by $\mathrm{Super}^{[n]}[\bar{\mathscr{L}}_k \mid \bar{\mathscr{P}}_{\mathbf{x}}]$ the $\mathbb{K}$-span of the elements

$$(\boldsymbol{\alpha}_1 \mid \mathbf{x})(\boldsymbol{\alpha}_1 \mid \mathbf{e}_{i_1}) \cdots (\boldsymbol{\alpha}_1 \mid \mathbf{e}_{i_k})(\boldsymbol{\alpha}_2 \mid \mathbf{x})(\boldsymbol{\alpha}_2 \mid \mathbf{e}_{j_1}) \cdots (\boldsymbol{\alpha}_2 \mid \mathbf{e}_{j_k}) \cdots$$
$$\cdots (\boldsymbol{\alpha}_n \mid \mathbf{x})(\boldsymbol{\alpha}_n \mid \mathbf{e}_{h_1}) \cdots (\boldsymbol{\alpha}_n \mid \mathbf{e}_{h_k})$$

in $\operatorname{Super}[\bar{\mathscr{L}} \mid \bar{\mathscr{P}}_{\mathbf{x}}]$. Define the $\mathbb{Z}_2$-*companion symbolic operators*

$$\bar{\mathbf{U}}\colon \operatorname{Super}^{[n]}[\bar{\mathscr{L}}_k \mid \bar{\mathscr{P}}_{\mathbf{x}^-}] \to \mathbf{S}^n(\mathbf{S}^k(V))$$

and

$$\bar{\mathbf{U}}'\colon \operatorname{Super}^{[n]}[\bar{\mathscr{L}}_k \mid \bar{\mathscr{P}}_{\mathbf{x}^+}] \to \Lambda^n(\mathbf{S}^k(V))$$

such that

$$\begin{aligned}\bar{\mathbf{U}}((\boldsymbol{\alpha}_1 \mid \mathbf{x}^-)(\boldsymbol{\alpha}_1 \mid \mathbf{e}_{i_1})\cdots(\boldsymbol{\alpha}_1 \mid \mathbf{e}_{i_k})\cdots(\boldsymbol{\alpha}_n \mid \mathbf{x}^-)(\boldsymbol{\alpha}_n \mid \mathbf{e}_{h_1})\cdots(\boldsymbol{\alpha}_n \mid \mathbf{e}_{h_k}))&\\ = [e_{i_1}\cdots e_{i_k}]\cdots[e_{h_1}\cdots e_{h_k}]& \qquad (8)\end{aligned}$$

and

$$\begin{aligned}\bar{\mathbf{U}}'((\boldsymbol{\alpha}_1 \mid \mathbf{x}^+)(\boldsymbol{\alpha}_1 \mid \mathbf{e}_{i_1})\cdots(\boldsymbol{\alpha}_1 \mid \mathbf{e}_{i_k})\cdots(\boldsymbol{\alpha}_n \mid \mathbf{x}^+)(\boldsymbol{\alpha}_n \mid \mathbf{e}_{h_1})\cdots(\boldsymbol{\alpha}_n \mid \mathbf{e}_{h_k}))&\\ = [e_{i_1}\cdots e_{i_k}]\cdots[e_{h_1}\cdots e_{h_k}].& \qquad (9)\end{aligned}$$

PROPOSITION 6. *The $\mathbb{Z}_2$-companion symbolic operators $\bar{\mathbf{U}}$ and $\bar{\mathbf{U}}'$ are well defined and surjective.*

The superalgebra $\operatorname{Super}[\bar{\mathscr{L}} \mid \bar{\mathscr{P}}_{\mathbf{x}}]$ is still a $\mathrm{pl}(V)$-module, where the action is similar to the one given in (7).

THEOREM 2. *Both $\bar{\mathbf{U}}$ and $\bar{\mathbf{U}}'$ are* $\mathrm{pl}(V)$-*equivariant linear surjective operators.*

Again the mock place $\mathbf{x}^-$ can be omitted in defining $\bar{\mathbf{U}}$, while $\mathbf{x}^+$ is crucial in defining $\bar{\mathbf{U}}'$.

The symmetric group S_n acts on $\operatorname{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}_x]$ (and $\operatorname{Super}^{[n]}[\bar{\mathscr{L}}_k \mid \bar{\mathscr{P}}_{\mathbf{x}}]$) linearly by setting

$$\begin{aligned}&\sigma\cdot((\alpha_1 \mid x)(\alpha_1 \mid e_{i_1})\cdots(\alpha_1 \mid e_{i_k})\cdots(\alpha_n \mid x)(\alpha_n \mid e_{h_1})\cdots(\alpha_n \mid e_{h_k}))\\ &\quad = (\alpha_{\sigma(1)} \mid x)(\alpha_{\sigma(1)} \mid e_{i_1})\cdots(\alpha_{\sigma(1)} \mid e_{i_k})\cdots(\alpha_{\sigma(n)} \mid x)(\alpha_{\sigma(n)} \mid e_{h_1})\cdots(\alpha_{\sigma(n)} \mid e_{h_k})\end{aligned}$$

for $\sigma \in S_n$.

PROPOSITION 7. *For every $\sigma \in S_n$, we have*

$$\begin{aligned}\mathbf{U}(\sigma(\mathbf{m})) &= \mathbf{U}(\mathbf{m}) && \textit{for all}\ \ \mathbf{m} \in \operatorname{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}_{x^+}],\\ \mathbf{U}'(\sigma(\mathbf{m})) &= \mathbf{U}'(\mathbf{m}) && \textit{for all}\ \ \mathbf{m} \in \operatorname{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}_{x^-}],\\ \bar{\mathbf{U}}(\sigma(\mathbf{m})) &= \bar{\mathbf{U}}(\mathbf{m}) && \textit{for all}\ \ \mathbf{m} \in \operatorname{Super}^{[n]}[\bar{\mathscr{L}}_k \mid \bar{\mathscr{P}}_{\mathbf{x}^-}],\\ \bar{\mathbf{U}}'(\sigma(\mathbf{m})) &= \bar{\mathbf{U}}'(\mathbf{m}) && \textit{for all}\ \ \mathbf{m} \in \operatorname{Super}^{[n]}[\bar{\mathscr{L}}_k \mid \bar{\mathscr{P}}_{\mathbf{x}^+}].\end{aligned}$$

Proof. To see this, we observe that the signatures of $(\alpha_i \mid x)(\alpha_i \mid e_{j_1}) \cdots (\alpha_i \mid e_{j_k})$ and $(\boldsymbol{\alpha}_i \mid \mathbf{x})(\boldsymbol{\alpha}_i \mid \mathbf{e}_{j_1}) \cdots (\boldsymbol{\alpha}_i \mid \mathbf{e}_{j_k})$ in the letterplace superalgebras agree with the signature of $[e_{j_1} e_{j_2} \cdots e_{j_k}]$ in the corresponding plethystic algebras. ▮

Remark. Although the symbolic operators are linear maps, they also possess a *multiplicative* property in the following sense. Let $\mathbf{U}_k$ denote, only in this paragraph, the symbolic operator from $\text{Super}^{[n]}[\mathcal{L}_k \mid \mathcal{P}_{x^+}]$ to $\mathbf{S}^n(\mathbf{S}^k(V))$. Suppose the sets of symbolic letters $\mathcal{L}_k$ and $\mathcal{L}_h$ are disjoint. Set $\mathcal{L}_{k+h} = \mathcal{L}_k \mathbin{\dot\cup} \mathcal{L}_h$. Let $\mathbf{m} \in \text{Super}^{[n]}[\mathcal{L}_k \mid \mathcal{P}_{x^+}]$ and $\mathbf{m}' \in \text{Super}^{[n]}[\mathcal{L}_h \mid \mathcal{P}_{x^+}]$. Then we have

$$\mathbf{U}_{k+h}(\mathbf{mm}') = \mathbf{U}_k(\mathbf{m})\, \mathbf{U}_h(\mathbf{m}').$$

A similar property is possessed by $\mathbf{U}'$, $\bar{\mathbf{U}}$, and $\bar{\mathbf{U}}'$.

3. Two Straightening Formulas for $\mathbf{S}^n(\mathbf{S}^2(V))$

In this section, we derive two different straightening formulas for $\mathbf{S}^n(\mathbf{S}^2(V))$ via the symbolic operators $\mathbf{U}$ and $\bar{\mathbf{U}}$. The straightening formula obtained by applying the operator $\mathbf{U}$ is the one of Rota and Stein for supersymmetric *Pfaffians* [47] which is a generalization of the straightening formula of De Concini and Procesi for invariants of the symplectic group [17]; the straightening formula obtained by applying the operator $\bar{\mathbf{U}}$ is the one of Rota and Stein for supersymmetric Hilbert spaces [48] which generalizes the straightening formula of De Concini and Procesi for invariants of the orthogonal group.

We will work with $\mathbf{U}$ and $\bar{\mathbf{U}}$ without using the mock place x, which is plausible as mentioned earlier. To every *even* partition λ, i.e., $\lambda = (2p, 2q, \cdots)$, we associate the *regular tableau* R_λ^+ on $\mathcal{L}$:

$$R_\lambda^+ = \begin{pmatrix} \alpha_1 & \alpha_1 & \alpha_2 & \alpha_2 & \cdots & \cdots & \alpha_p & \alpha_p \\ \alpha_{p+1} & \alpha_{p+1} & \cdots & \cdots & \alpha_{p+q} & \alpha_{p+q} & & \\ \cdots & \cdots & & & & & & \end{pmatrix}.$$

If $\tilde{\lambda}$, the conjugate of λ, is even, we define the *regular tableau* $\mathbf{R}_\lambda^-$ on $\bar{\mathcal{L}}$ to be the conjugate of $R_{\tilde{\lambda}}^+$. To save words, by a *tableau* S on $\mathcal{L}_2$ (or $\mathbf{S}$ *on* $\bar{\mathcal{L}}_2$) we mean a tableau such that each letter α_i (or $\boldsymbol{\alpha}_i$), $i = 1, 2, ..., n$, occurs twice; and by a shape λ we mean a shape such that $\lambda \vdash 2n$, when working on $\mathbf{S}^n(\mathbf{S}^2(V))$ or on $\Lambda^n(\mathbf{S}^2(V))$. When no confusion arises, we write $\mathbf{U}((S \mid T))$ as $\mathbf{U} \cdot (S \mid T)$ or simply as $\mathbf{U}(S \mid T)$.

In the present paper, we always use the *dominance order* among the shapes.

PROPOSITION 8 (Regularization Algorithms for $\mathbf{S}^n(\mathbf{S}^2(V))$). *For every pair of tableaux S on $\mathscr{L}_2$ and T on $\mathscr{P}$, we have*

$$\mathbf{U}\cdot(S\,|\,T)=\sum_{\substack{\lambda\ even\\ \lambda\geqslant \mathrm{sh}(S)}}\sum_{Q} c_{\lambda Q}\mathbf{U}\cdot(R_\lambda^+\mid Q)\ in\ \mathbf{S}^n(\mathbf{S}^2(V)), \tag{10}$$

where each Q is a tableau on $\mathscr{P}$ of shape λ and $c_{\lambda Q}\in\mathbb{K}$. Similarly, for every pair of tableaux $\mathbf{S}$ *on* $\bar{\mathscr{L}}_2$ *and* $\mathbf{T}$ *on* $\bar{\mathscr{P}}$, *we have*

$$\bar{\mathbf{U}}\cdot(\mathbf{S}\mid\mathbf{T})=\sum_{\substack{\tilde{\lambda}\ even\\ \lambda\geqslant \mathrm{sh}(\mathbf{S})}}\sum_{\mathbf{Q}} d_{\lambda\mathbf{Q}}\,\bar{\mathbf{U}}\cdot(\mathbf{R}_\lambda^-\mid\mathbf{Q})\ in\ \mathbf{S}^n(\mathbf{S}^2(V)). \tag{11}$$

Proof. To prove (10), by Proposition 7, it is sufficient to show that

$$\mathbf{U}\cdot(S\mid T)=\sum_{R,\,Q} c_{RQ}\mathbf{U}\cdot(R\mid Q), \tag{12}$$

where $R=\sigma(R_\lambda^+)$ for some $\sigma\in S_n$ and some even $\lambda\geqslant\mathrm{sh}(S)$. That is, for each letter α_i, the two α_i's always appear consecutively in the same row of R. By sorting within the rows, we can assume that every two α_i's either appear consecutively in some row or appear in different rows of S. Order the positions in the Ferrers diagram of $\mathrm{sh}(S)$ such that the (i,j)-position is before the (i',j')-position if either $i<i'$ or $i=i'$ and $j<j'$. If S is not of the form $\sigma(R_\lambda^+)$, let (i,j) be the first position in λ such that the letter, say α_h, occupying the position is not followed by the identical letter in the same row. Then the other α_h must appear in a later row in S. Without loss of generality, we can assume that S has two rows, say

$$S=\begin{pmatrix}\alpha_1\ \alpha_1\ \cdots\ \alpha_{h-1}\ \alpha_{h-1}\ \alpha_h\ v\\ \alpha_h\ w\end{pmatrix}\qquad\text{and}\qquad T=\begin{pmatrix}v'\\ w'\end{pmatrix},$$

where $v,\ w\in\mathrm{Mon}(\mathscr{L})$ and $v',\ w'\in\mathrm{Mon}(\mathscr{P})$. Applying the exchange identity in [23, Proposition 10], we obtain

$$\begin{aligned}
&\sum_{i=1}^{h} 2\left(\begin{matrix}\alpha_1\ \alpha_1\ \cdots\ \alpha_{i-1}\ \alpha_{i-1}\ \alpha_i\ \alpha_{i+1}\ \alpha_{i+1}\ \cdots\ \alpha_h\ \alpha_h\ v\\ \alpha_i\ w\end{matrix}\ \middle|\ \begin{matrix}v'\\ w'\end{matrix}\right)\\
&\qquad=-\sum_{v}\left(\begin{matrix}\alpha_1\ \ \alpha_1\ \cdots\ \alpha_h\ \alpha_h\ v_{(1)}\\ v_{(2)}\ \ w\end{matrix}\ \middle|\ \begin{matrix}v'\\ w'\end{matrix}\right)\\
&\qquad\quad+\sum_{\substack{w'\\ \mathrm{length}(w'_{(1)})=1}}\left(\begin{matrix}\alpha_1\ \alpha_1\ \cdots\ \alpha_h\ \alpha_h\ v\\ w\end{matrix}\ \middle|\ \begin{matrix}v'\quad w'_{(1)}\\ w'_{(2)}\end{matrix}\right).
\end{aligned}\tag{13}$$

Note that (i) each of the h terms on the left has the same image as $2\mathbf{U}\cdot(S \mid T)$; (ii) for a term in the first sum on the right side, if a letter is not followed by the identical letter in the same row, this will occupy a later position in S than α_h; (iii) each term in the second sum on the right side has a shape longer than $\mathrm{sh}(S)$. Hence, by induction, (12) can be achieved and so can (10). The idea to prove (11) is similar since one can obtain by the exchange identity that

$$\sum_i (-1)^{h-i}\left(\begin{array}{cccccc|c} \boldsymbol{\alpha}_1 & \boldsymbol{\alpha}_2 & \cdots\cdots & & \boldsymbol{\alpha}_h & \mathbf{u} & \mathbf{u}' \\ \boldsymbol{\alpha}_1 & \cdots & \hat{\boldsymbol{\alpha}}_i & \cdots & \boldsymbol{\alpha}_h & \mathbf{v} & \mathbf{v}' \\ \boldsymbol{\alpha}_i & \mathbf{w} & & & & & \mathbf{w}' \end{array}\right)$$

$$= \sum_{\mathbf{v},\mathbf{w}'} \pm \left(\begin{array}{ccccc|cc} \boldsymbol{\alpha}_1 & \boldsymbol{\alpha}_2 & \cdots & \boldsymbol{\alpha}_h & \mathbf{u} & \mathbf{u}' & \\ \boldsymbol{\alpha}_1 & \boldsymbol{\alpha}_2 & \cdots & \boldsymbol{\alpha}_h & \mathbf{v}_{(1)} & \mathbf{v}' & \mathbf{w}'_{(1)} \\ \mathbf{v}_{(2)} & \mathbf{w} & & & & \mathbf{w}'_{(2)} & \end{array}\right).$$

The terms (including the front signs) on the left have the same image under $\bar{\mathbf{U}}$ and formula (11) follows from similar arguments. ∎

THEOREM 3 (Rota and Stein). *An element* $\mathbf{m} \in \mathbf{S}^n(\mathbf{S}^2(V))$ *can be uniquely written both as*

$$\mathbf{m} = \sum_{\substack{\lambda \vdash 2n \\ \lambda \text{ even}}} \sum_{\substack{T \text{ standard} \\ \text{on } \mathscr{P}}} c_{\lambda T}\mathbf{U}\cdot(R_\lambda^+ \mid T) \tag{14}$$

and as

$$\mathbf{m} = \sum_{\substack{\lambda \vdash 2n \\ \lambda \text{ even}}} \sum_{\substack{\mathbf{T} \text{ standard} \\ \text{on } \mathscr{P}}} d_{\lambda \mathbf{T}}\bar{\mathbf{U}}\cdot(\mathbf{R}_\lambda^- \mid \mathbf{T}). \tag{15}$$

Proof. The spanning part of the theorem follows from combining the previous regularization algorithms with the straightening formula for $\mathrm{Super}[\mathscr{L} \mid \mathscr{P}]$ in [23].

The uniqueness part follows immediately from the Gordan–Capelli series for $\mathbf{S}^n(\mathbf{S}^2(V))$ to be discussed later in the next section. ∎

Formulas (14) and (15) give two different straightening formulas for $\mathbf{S}^n(\mathbf{S}^2(V))$. In fact, (14) is the one for supersymmetric Pfaffians of Rota and Stein [47]; while (15) is the straightening formula for supersymmetric Hilbert spaces in [48]. To make these facts apparent, let $V = V_1$, so $\mathscr{P} = \mathscr{P}^-$. We have

$$\begin{aligned}
&\mathbf{U}\cdot(\alpha_1\alpha_1\alpha_2\alpha_2\cdots\alpha_p\alpha_p \mid e_1e_2e_3e_4\cdots e_{2p-1}e_{2p})\\
&\quad=\mathbf{U}\cdot \sum_{\substack{\sigma\in S_{2p}\\ \sigma(2i-1)<\sigma(2i)}} (-1)^{|\sigma|}(\alpha_1\alpha_1 \mid e_{\sigma(1)}e_{\sigma(2)})\\
&\qquad\times(\alpha_2\alpha_2 \mid e_{\sigma(3)}e_{\sigma(4)})\cdots(\alpha_p\alpha_p \mid e_{\sigma(2p-1)}e_{\sigma(2p)})\\
&\quad=\sum_{\substack{\sigma\in S_{2p}\\ \sigma(2i-1)<\sigma(2i)}} (-1)^{|\sigma|}\,2^p[e_{\sigma(1)}e_{\sigma(2)}][e_{\sigma(3)}e_{\sigma(4)}]\cdots[e_{\sigma(2p-1)}e_{\sigma(2p)}]\\
&\quad=p!\,2^p\,\mathrm{Pfaffian}([e_ie_j]_{1\leqslant i,j\leqslant 2p}).
\end{aligned}$$

In general,

$$\mathbf{U}\cdot(\alpha_1\alpha_1\alpha_2\alpha_2\cdots\alpha_p\alpha_p \mid e_{i_1}e_{i_2}e_{i_3}e_{i_4}\cdots e_{i_{2p-1}}e_{i_{2p}})=p!\,2^p\,\mathrm{Pfaffian}(M_{i_1i_2\cdots i_{2p}}),$$

where $M_{i_1\cdots i_{2p}}$ is the submatrix of the skew-symmetric matrix $([e_ie_j]_{1\leqslant i,j\leqslant d})$ consisting of rows $i_1, i_2, ..., i_{2p}$ and columns $i_1, i_2, ..., i_{2p}$.

Pfaffian $(M_{i_1i_2\cdots i_{2p}})$ is also called a *partial Pfaffian* associated with the submatrix $M_{i_1i_2\cdots i_{2p}}$. Therefore $\mathbf{U}\cdot(R_\lambda^+ \mid T)$, which equals the product of the images of the rows of $(R_\lambda^+ \mid T)$, is a product of partial Pfaffians.

If V is $\mathbb{Z}_2$-graded, we get the *supersymmetric Pfaffian*

$$\mathbf{U}\cdot(\alpha_1\alpha_1\alpha_2\alpha_2\cdots\alpha_p\alpha_p \mid e_{i_1}e_{i_2}\cdots e_{i_{2p-1}}e_{i_{2p}})=p!\,2^p\,\mathrm{Pfaffian}(e_{i_1}e_{i_2}\cdots e_{i_{2p-1}}e_{i_{2p}}),$$

in the notation of Rota and Stein.

Next, let us compute

$$\mathbf{U}\cdot\begin{pmatrix}\alpha_1 & \alpha_2 & \cdots & \alpha_n & e_{i_1} & e_{i_2} & \cdots & e_{i_n}\\ \alpha_1 & \alpha_2 & \cdots & \alpha_n & e_{j_1} & e_{j_2} & \cdots & e_{j_n}\end{pmatrix}.$$

Suppose $\mathscr{P}$ is large enough and let $y^+, z^+ \in \mathscr{P}\setminus\{e_{i_1}, ..., e_{i_n}, e_{j_1}, ..., e_{j_n}\}$. Then

$$\begin{aligned}
&\mathbf{U}\cdot\begin{pmatrix}\alpha_1 & \alpha_2 & \cdots & \alpha_n & e_{i_1} & e_{i_2} & \cdots & e_{i_n}\\ \alpha_1 & \alpha_2 & \cdots & \alpha_n & e_{j_1} & e_{j_2} & \cdots & e_{j_n}\end{pmatrix}\\
&\quad=\mathbf{U}\cdot\left(\begin{pmatrix}\alpha_1 & \alpha_2 & \cdots & \alpha_n & y^{(n)}\\ \alpha_1 & \alpha_2 & \cdots & \alpha_n & z^{(n)}\end{pmatrix}{}_{ye_{i_1}}\mathcal{a}\cdots{}_{ye_{i_n}}\mathcal{a}\,{}_{ze_{j_1}}\mathcal{a}\cdots{}_{ze_{j_n}}\mathcal{a}\right)\\
&\quad=\left(\mathbf{U}\cdot\begin{pmatrix}\alpha_1 & \alpha_2 & \cdots & \alpha_n & y^{(n)}\\ \alpha_1 & \alpha_2 & \cdots & \alpha_n & z^{(n)}\end{pmatrix}\right){}_{ye_{i_1}}\mathcal{a}\cdots{}_{ye_{i_n}}\mathcal{a}\,{}_{ze_{j_1}}\mathcal{a}\cdots{}_{ze_{j_n}}\mathcal{a},
\end{aligned}$$

since $\mathbf{U}$ is $\mathrm{pl}(V)$-equivariant. The last expression equals

$$\begin{aligned}
&[yz]^n\,{}_{ye_{i_1}}\mathcal{a}\cdots{}_{ye_{i_n}}\mathcal{a}\,{}_{ze_{j_1}}\mathcal{a}\cdots{}_{ze_{j_n}}\mathcal{a}\\
&\quad=n!\,([e_{i_1}z]\cdots[e_{i_n}z])\,{}_{ze_{j_1}}\mathcal{a}\cdots{}_{ze_{j_n}}\mathcal{a}\\
&\quad=n!\,(e_{i_1}e_{i_2}\cdots e_{i_n} \mid e_{j_1}e_{j_2}\cdots e_{j_n}),
\end{aligned}$$

which is the biproduct in [23] of the supersymmetric matrix $([e_{i_s}e_{j_t}]_{1 \leqslant s, t \leqslant n})$. If $|e_i| = 1$, $i = 1, 2, ..., d$, then $(e_{i_1}e_{i_2} \cdots e_{i_n} \mid e_{j_1}e_{j_2} \cdots e_{j_n})$ is, up to the sign $(-1)^{n(n-1)/2}$, the determinant of the submatrix of rows $i_1, i_2, ..., i_n$ and columns $j_1, j_2, ..., j_n$ of the skew-symmetric matrix $([e_ie_j]_{1 \leqslant i,j \leqslant d})$. On the other hand, if $|e_i| = 0$, $i = 1, 2, ..., d$, then $(e_{i_1}e_{i_2} \cdots e_{i_n} \mid e_{j_1}e_{j_2} \cdots e_{j_n})$ is the permanent of the same submatrix of the symmetric matrix $([e_ie_j]_{1 \leqslant i,j \leqslant d})$.

Restricting to the case $V = V_1$, we have

$$\mathbf{U} \cdot \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_d & e_1 & e_2 & \cdots & e_d \\ \alpha_1 & \alpha_2 & \cdots & \alpha_d & e_1 & e_2 & \cdots & e_d \end{pmatrix} = (-1)^{d(d-1)/2}\, d!\, \det([e_ie_j]_{1 \leqslant i,j \leqslant d}).$$

On the other hand, by the regularization algorithm,

$$\mathbf{U} \cdot \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_d & e_1 & e_2 & \cdots & e_d \\ \alpha_1 & \alpha_2 & \cdots & \alpha_d & e_1 & e_2 & \cdots & e_d \end{pmatrix}$$
$$= c\mathbf{U} \cdot \begin{pmatrix} \alpha_1 & \alpha_1 & \cdots \alpha_p & \alpha_p & e_1 & e_2 \cdots e_d \\ \alpha_{p+1} & \alpha_{p+1} & \cdots \alpha_{2p} & \alpha_{2p} & e_2 & e_2 \cdots e_d \end{pmatrix},$$

for some $c \in \mathbb{K}$, if $d = 2p$ is even; it is zero if d is odd. To find out the scalar c, let us compare the coefficients of

$$[e_1e_2][e_2e_1][e_3e_4][e_4e_3] \cdots [e_{2p-1}e_{2p}][e_{2p}e_{2p-1}]$$
$$= (-1)^p [e_1e_2]^2 \cdots [e_{2p-1}e_{2p}]^2$$

on both sides of the above identity. Clearly, its coefficient in $\det([e_ie_j]_{1 \leqslant i,j \leqslant d})$ is $(-1)^p = (-1)^{d(d-1)/2}$. On the other hand, the coefficient of $[e_1e_2] \cdots [e_{2p-1}e_{2p}]$ in $\mathbf{U} \cdot (\alpha_1\alpha_1 \cdots \alpha_p\alpha_p \mid e_1e_2 \cdots e_{2p-1}e_{2p})$ is $2^p p!$. Therefore

$$c = (-1)^p \frac{(2p)!}{(p!)^2\, 2^{2p}}.$$

Recalling that

$$\mathbf{U} \cdot (\alpha_1\alpha_1 \cdots \alpha_p\alpha_p \mid e_1e_2 \cdots e_{2p-1}e_{2p}) = p!\, 2^p \,\text{Pfaffian}([e_ie_j]_{1 \leqslant i,j \leqslant 2p}),$$

we get

$$\det([e_ie_j]_{1 \leqslant i,j \leqslant d}) = \text{Pfaffian}^2([e_ie_j]_{1 \leqslant i,j \leqslant d}),$$

a well-known classical result.

At this point it is interesting to note that one can have a notion of Pfaffian even for symmetric matrices, since this corresponds to the case $V = V_0$. In fact, such a Pfaffian should be given by

$$\frac{1}{2^p p!}\mathbf{U}\cdot(\alpha_1\alpha_1\alpha_2\alpha_2\cdots\alpha_p\alpha_p \mid e_1e_2\cdots e_{2p-1}e_{2p})$$
$$=\frac{1}{2^p p!}\mathbf{U}\cdot \sum_{\substack{\sigma\in S_{2p}\\ \sigma(2i-1)<\sigma(2i)}} (\alpha_1\alpha_1 \mid e_{\sigma(1)}e_{\sigma(2)})\cdots(\alpha_p\alpha_p \mid e_{\sigma(2p-1)}e_{\sigma(2p)})$$
$$=\frac{1}{p!}\sum_{\substack{\sigma\in S_{2p}\\ \sigma(2i-1)<\sigma(2i)}} [e_{\sigma(1)}e_{\sigma(2)}][e_{\sigma(3)}e_{\sigma(4)}]\cdots[e_{\sigma(2p-1)}e_{\sigma(2p)}].$$

One can check that the square of this expression is the permanent of the symmetric matrix $([e_ie_j]_{1\leqslant i,j\leqslant 2p})$ not taking into account the terms involving diagonal entries. In general, the square of the supersymmetric Pfaffian$(e_1e_2\cdots e_{2p})$ equals, up to a sign, the biproduct $(e_1e_2\cdots e_{2p} \mid e_1e_2\cdots e_{2p})$ of the supersymmetric matrix $([e_ie_j]_{1\leqslant i,j\leqslant 2p})$ not taking into account the terms involving diagonal entries. This assertion can be proved by applying the regularization algorithm to

$$\mathbf{U}\cdot\begin{pmatrix}\alpha_1 & \alpha_2\cdots\alpha_{2p} & e_1 & e_2\cdots e_{2p}\\ \alpha_1 & \alpha_2\cdots\alpha_{2p} & e_1 & e_2\cdots e_{2p}\end{pmatrix}.$$

The notion of *standardness* for a bitableau $(R_\lambda^+ \mid T)$ also agrees with the one of Rota and Stein for supersymmetric Pfaffians. To see this, let λ be an even shape $(2p, 2q, ...)$ and T a tableau $(\omega_1, \omega_2, ...)$ of shape λ; then

$$\mathbf{U}\cdot(R_\lambda^+ \mid T)=\mathbf{U}\cdot(\alpha_1\alpha_1\cdots\alpha_p\alpha_p \mid \omega_1)\,\mathbf{U}\cdot(\alpha_{p+1}\alpha_{p+1}\cdots\alpha_{p+q}\alpha_{p+q} \mid \omega_2)\cdots$$
$$=c\cdot\text{Pfaffian}(\omega_1)\,\text{Pfaffian}(\omega_2)\cdots=c\cdot\text{Pfaffian}(T),\qquad c\in\mathbb{K},$$

in the notation of Rota and Stein.

Next, we discuss the relation between formula (15) and the straightening formula of Rota and Stein for supersymmetric Hilbert spaces.

First, let us compute

$$\bar{\mathbf{U}}\cdot\begin{pmatrix}\boldsymbol{\alpha}_1 & \boldsymbol{\alpha}_2\cdots\boldsymbol{\alpha}_n & \mathbf{e}_{i_1} & \mathbf{e}_{i_2}\cdots\mathbf{e}_{i_n}\\ \boldsymbol{\alpha}_1 & \boldsymbol{\alpha}_2\cdots\boldsymbol{\alpha}_n & \mathbf{e}_{j_1} & \mathbf{e}_{j_2}\cdots\mathbf{e}_{j_n}\end{pmatrix}.$$

Suppose $\mathscr{P}$ is large enough and let $y^-, z^-\in\mathscr{P}\backslash\{e_{i_1}, ..., e_{i_n}, e_{j_1}, ..., e_{j_n}\}$. Noting that $\mathbf{y}$ and $\mathbf{z}$ are positive in $\bar{\mathscr{P}}$, we have

$$\bar{\mathbf{U}}\cdot\begin{pmatrix}\boldsymbol{\alpha}_1 & \boldsymbol{\alpha}_2 & \cdots & \boldsymbol{\alpha}_n & \mathbf{e}_{i_1} & \mathbf{e}_{i_2} & \cdots & \mathbf{e}_{i_n}\\ \boldsymbol{\alpha}_1 & \boldsymbol{\alpha}_2 & \cdots & \boldsymbol{\alpha}_n & \mathbf{e}_{j_1} & \mathbf{e}_{j_2} & \cdots & \mathbf{e}_{j_n}\end{pmatrix}$$
$$=\bar{\mathbf{U}}\cdot\left(\begin{pmatrix}\boldsymbol{\alpha}_1 & \boldsymbol{\alpha}_2 & \cdots & \boldsymbol{\alpha}_n & \mathbf{y}^{(n)}\\ \boldsymbol{\alpha}_1 & \boldsymbol{\alpha}_2 & \cdots & \boldsymbol{\alpha}_n & \mathbf{z}^{(n)}\end{pmatrix}{}_{\mathbf{y}e_{i_1}}\mathcal{D}\cdots{}_{\mathbf{y}e_{i_n}}\mathcal{D}\,{}_{\mathbf{z}e_{j_1}}\mathcal{D}\cdots{}_{\mathbf{z}e_{j_n}}\mathcal{D}\right)$$
$$=\left(\bar{\mathbf{U}}\cdot\begin{pmatrix}\boldsymbol{\alpha}_1 & \boldsymbol{\alpha}_2 & \cdots & \boldsymbol{\alpha}_n & \mathbf{y}^{(n)}\\ \boldsymbol{\alpha}_1 & \boldsymbol{\alpha}_2 & \cdots & \boldsymbol{\alpha}_n & \mathbf{z}^{(n)}\end{pmatrix}\right){}_{ye_{i_1}}\mathcal{D}\cdots{}_{ye_{i_n}}\mathcal{D}\,{}_{ze_{j_1}}\mathcal{D}\cdots{}_{ze_{j_n}}\mathcal{D},$$

since $\tilde{\mathbf{U}}$ is pl(V)-equivariant. The last expression equals

$$\begin{aligned}&(-1)^{n(n-1)/2}\,[y^- z^-]^n\,{}_{ye_{i_1}}\mathcal{a}\cdots{}_{ye_{i_n}}\mathcal{a}\,{}_{ze_{j_1}}\mathcal{a}\cdots{}_{ze_{j_n}}\mathcal{a}\\ &=n!\,(-1)^{n(n-1)/2}\,(-1)^{|e_{i_1}|+\cdots+|e_{i_n}|+n}\\ &\quad\times([e_{i_1}z^-]\cdots[e_{i_n}z^-])\,{}_{ze_{j_1}}\mathcal{a}\cdots{}_{ze_{j_n}}\mathcal{a}\end{aligned}$$

Since $|{}_{ze_{j_1}}\mathcal{a}|=|e_{j_1}|+1$, the last expression is equal to

$$n!\,(-1)^{n(n+1)/2+|e_{i_1}|+\cdots+|e_{i_n}|}\,(e_{i_1}e_{i_2}\cdots e_{i_n}\mid e_{j_1}e_{j_2}\cdots e_{j_n})^*,$$

where the *-biproduct is defined (see [8]) such that

$$(\omega\mid\omega')^*=(\boldsymbol{\omega}\mid\boldsymbol{\omega}');$$

that is, every variable changes its signature.

The object $(e_{i_1}e_{i_2}\cdots e_{i_n}\mid e_{j_1}e_{j_2}\cdots e_{j_n})^*$, called the *supersymmetric Gramian* of the supersymmetric matrix $([e_{i_s}e_{j_t}]_{1\leqslant s,t\leqslant n})$, is studied in [48] and is denoted there as $(\mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_n}\mid\mathbf{e}_{j_1}\mathbf{e}_{j_2}\cdots\mathbf{e}_{j_n})$.

If $|e_i|=1$, $i=1,2,\ldots,d$, then

$$\begin{aligned}&\tilde{\mathbf{U}}\cdot\left(\begin{array}{ccc|ccc}\boldsymbol{\alpha}_1&\boldsymbol{\alpha}_2\cdots&\boldsymbol{\alpha}_n&\mathbf{e}_{i_1}&\mathbf{e}_{i_2}\cdots&\mathbf{e}_{i_n}\\ \boldsymbol{\alpha}_1&\boldsymbol{\alpha}_2\cdots&\boldsymbol{\alpha}_n&\mathbf{e}_{j_1}&\mathbf{e}_{j_2}\cdots&\mathbf{e}_{j_n}\end{array}\right)\\ &\quad=n!\,(-1)^{n(n-1)/2}\,(e_{i_1}e_{i_2}\cdots e_{i_n}\mid e_{j_1}e_{j_2}\cdots e_{j_n})^*\\ &\quad=n!\,(-1)^{n(n-1)/2}\,\mathrm{per}([e_{i_s}e_{j_t}]_{1\leqslant s,t\leqslant n}).\end{aligned}$$

If $|e_i|=0$, $i=1,2,\ldots,d$, then

$$\begin{aligned}&\tilde{\mathbf{U}}\cdot\left(\begin{array}{ccc|ccc}\boldsymbol{\alpha}_1&\boldsymbol{\alpha}_2\cdots&\boldsymbol{\alpha}_n&\mathbf{e}_{i_1}&\mathbf{e}_{i_2}\cdots&\mathbf{e}_{i_n}\\ \boldsymbol{\alpha}_1&\boldsymbol{\alpha}_2\cdots&\boldsymbol{\alpha}_n&\mathbf{e}_{j_1}&\mathbf{e}_{j_2}\cdots&\mathbf{e}_{j_n}\end{array}\right)\\ &\quad=n!\,(-1)^{n(n+1)/2}\,(e_{i_1}e_{i_2}\cdots e_{i_n}\mid e_{j_1}e_{j_2}\cdots e_{j_n})^*\\ &\quad=n!\,(-1)^{n}\,\det([e_{i_s}e_{j_t}]_{1\leqslant s,t\leqslant n}).\end{aligned}$$

The determinant $\det([e_{i_s}e_{j_t}]_{1\leqslant s,t\leqslant n})$ is called the Gramian of the symmetric matrix $([e_ie_j]_{1\leqslant i,j\leqslant d})$ associated to the submatrix of rows $i_1,i_2,\ldots,i_n$ and columns $j_1,j_2,\ldots,j_n$. This Gramian is denoted as $(\mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_n}\mid\mathbf{e}_{j_1}\mathbf{e}_{j_2}\cdots\mathbf{e}_{j_n})$ in [17].

In fact, the last formula can be derived directly as follows:

$$\begin{aligned}&\tilde{\mathbf{U}}\cdot\left(\begin{array}{ccc|ccc}\boldsymbol{\alpha}_1&\boldsymbol{\alpha}_2\cdots&\boldsymbol{\alpha}_n&\mathbf{e}^-_{i_1}&\mathbf{e}^-_{i_2}\cdots&\mathbf{e}^-_{i_n}\\ \boldsymbol{\alpha}_1&\boldsymbol{\alpha}_2\cdots&\boldsymbol{\alpha}_n&\mathbf{e}^-_{j_1}&\mathbf{e}^-_{j_2}\cdots&\mathbf{e}^-_{j_n}\end{array}\right)\\ &\quad=(-1)^n\,\tilde{\mathbf{U}}\cdot(\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2\cdots\boldsymbol{\alpha}_n\mid\mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_n})\,\tilde{\mathbf{U}}\cdot(\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2\cdots\boldsymbol{\alpha}_n\mid\mathbf{e}_{j_1}\mathbf{e}_{j_2}\cdots\mathbf{e}_{j_n})\end{aligned}$$

$$= (-1)^n \bar{\mathbf{U}} \cdot \sum_{\sigma, \tau \in S_n} (-1)^{|\sigma| + |\tau|} (\boldsymbol{\alpha}_1 \mid \mathbf{e}_{i_{\sigma(1)}}) \cdots (\boldsymbol{\alpha}_n \mid \mathbf{e}_{i_{\sigma(n)}})$$

$$\times (\boldsymbol{\alpha}_1 \mid \mathbf{e}_{j_{\tau(1)}}) \cdots (\boldsymbol{\alpha}_n \mid \mathbf{e}_{j_{\tau(n)}})$$

$$= (-1)^n \sum_{\sigma, \tau \in S_n} (-1)^{|\sigma| + |\tau|} [e_{i_{\sigma(1)}} e_{j_{\tau(1)}}] \cdots [e_{i_{\sigma(n)}} e_{j_{\tau(n)}}]$$

$$= (-1)^n \sum_{\sigma \in S_n} (-1)^{|\sigma|} \det([e_{i_{\sigma(s)}} e_{j_t}]_{1 \leqslant s, t \leqslant n})$$

$$= (-1)^n n! \det([e_{i_s} e_{j_t}]_{1 \leqslant s, t \leqslant n}).$$

For a general tableau T of shape $\lambda = (p, p, q, q, ...)$, we have

$$\bar{\mathbf{U}} \cdot (\mathbf{R}_\lambda^- \mid \mathbf{T}) = \bar{\mathbf{U}} \cdot \left(\begin{array}{cccc|c} \boldsymbol{\alpha}_1 & \boldsymbol{\alpha}_2 & \cdots & \boldsymbol{\alpha}_p & \boldsymbol{\omega}_1 \\ \boldsymbol{\alpha}_1 & \boldsymbol{\alpha}_2 & \cdots & \boldsymbol{\alpha}_p & \boldsymbol{\omega}_1' \\ \boldsymbol{\alpha}_{p+1} & \boldsymbol{\alpha}_{p+2} & \cdots & \boldsymbol{\alpha}_{p+q} & \boldsymbol{\omega}_2 \\ \boldsymbol{\alpha}_{p+1} & \boldsymbol{\alpha}_{p+2} & \cdots & \boldsymbol{\alpha}_{p+q} & \boldsymbol{\omega}_2' \\ \cdots & & & & \cdots \end{array} \right)$$

$$= \pm \bar{\mathbf{U}} \cdot \left(\begin{array}{cccc|c} \boldsymbol{\alpha}_1 & \boldsymbol{\alpha}_2 & \cdots & \boldsymbol{\alpha}_p & \boldsymbol{\omega}_1 \\ \boldsymbol{\alpha}_1 & \boldsymbol{\alpha}_2 & \cdots & \boldsymbol{\alpha}_p & \boldsymbol{\omega}_1' \end{array} \right) \bar{\mathbf{U}} \cdot \left(\begin{array}{cccc|c} \boldsymbol{\alpha}_{p+1} & \boldsymbol{\alpha}_{p+2} & \cdots & \boldsymbol{\alpha}_{p+q} & \boldsymbol{\omega}_2 \\ \boldsymbol{\alpha}_{p+1} & \boldsymbol{\alpha}_{p+2} & \cdots & \boldsymbol{\alpha}_{p+q} & \boldsymbol{\omega}_2' \end{array} \right) \cdots$$

$$= c(\omega_1 \mid \omega_1')^* \, (\omega_2 \mid \omega_2')^* \cdots = c(\boldsymbol{\omega}_1 \mid \boldsymbol{\omega}_1')(\boldsymbol{\omega}_2 \mid \boldsymbol{\omega}_2') \cdots,$$

for some $c \in \mathbb{K}$.

The above formula explains the seemingly odd notion of standard bitableaux of De Concini and Procesi [17, p. 344] and Rota and Stein [48, p. 656] regarding Gramians. To remind the readers, in their notation, a bitableau

$$\left(\begin{array}{c|c} \boldsymbol{\omega}_1 & \boldsymbol{\omega}_1' \\ \boldsymbol{\omega}_2 & \boldsymbol{\omega}_2' \\ \cdots & \cdots \end{array} \right)$$

is called *standard* if the tableau

$$\left(\begin{array}{c} \boldsymbol{\omega}_1 \\ \boldsymbol{\omega}_1' \\ \boldsymbol{\omega}_2 \\ \boldsymbol{\omega}_2' \\ \cdots \end{array} \right)$$

is standard in the ordinary sense.

Now let ω, ω_1, ω_2 be three words in Mon($\mathscr{P}$) such that length(ω) > n and length(ω) + length(ω_1) + length(ω_2) = $2n$. We have

$$\sum_{\boldsymbol{\omega}} \left(\begin{array}{ccc|cc} \boldsymbol{\alpha}_1 & \boldsymbol{\alpha}_2 \cdots & \boldsymbol{\alpha}_n & \boldsymbol{\omega}_1 & \boldsymbol{\omega}_{(1)} \\ \boldsymbol{\alpha}_1 & \boldsymbol{\alpha}_2 \cdots & \boldsymbol{\alpha}_n & \boldsymbol{\omega}_{(2)} & \boldsymbol{\omega}_2 \end{array} \right) = 0 \quad \text{in} \quad \mathrm{Super}[\bar{\mathscr{L}} \mid \bar{\mathscr{P}}], \tag{16}$$

where $\Delta\boldsymbol{\omega} = \sum_{\boldsymbol{\omega}} \boldsymbol{\omega}_{(1)} \otimes \boldsymbol{\omega}_{(2)}$ is the coproduct in the Hopf algebra $\mathbf{S}(\bar{V})$. Denote by $\sum_\omega \omega_{[1]} \otimes \omega_{[2]}$ the corresponding expression in $\mathbf{S}(V)$. Applying the symbolic operator $\mathbf{U}$ to (16), we get

$$\sum_\omega (\omega_1 \omega_{[1]} \mid \omega_{[2]} \omega_2)^* = 0,$$

the basic identity which holds for Gramians of a supersymmetric matrix. This identity is the syntactic superalgebraic version of Lemma 5.2 of [17].

4. Two Gordan–Capelli Series for $\mathbf{S}^n(\mathbf{S}^2(V))$

In this section we describe two different Gordan–Capelli series for $\mathbf{S}^n(\mathbf{S}^2(V))$ via the symbolic operators $\mathbf{U}$ and $\bar{\mathbf{U}}$. In addition to the fact that they provide a complete decomposition of $\mathbf{S}^n(\mathbf{S}^2(V))$ into pl(V)-irriducibles, they also give transparent proofs of the linear independence of standard tableaux of supersymmetric Pfaffians and Gramians.

Without loss of generality, we assume that $\mathscr{P}$ has a sufficiently large supply of both negative and positive symbols. Under this assumption, the Gordan–Capelli series to be derived holds in general except that, for a given V, some isotypic components simply disappear depending on $d_0 = \dim(V_0)$ and $d_1 = \dim(V_1)$. To be more precise, an even shape $\lambda \vdash 2n$ occurs later in (19) only when $\lambda_{d_0+1} \leqslant d_1$; a shape $\lambda \vdash 2n$, such that $\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, ...)$ is even, occurs later in (21) only when $\lambda_{d_1+1} \leqslant d_0$. These are the so-called *hook conditions* of Berele and Regev (see [6, 8]).

Applying the general results of [8, 9] to the Gordan–Capelli series of Super[$\mathscr{L} \mid \mathscr{P}$], we get

$$\mathrm{Super}^{[n]}[\mathscr{L}_2 \mid \mathscr{P}] = \bigoplus_{\lambda \vdash 2n} \bigoplus_{\substack{S \text{ standard on } \mathscr{L}_2 \\ \mathrm{sh}(S) = \lambda}} \mathfrak{S}_{\lambda S}, \tag{17}$$

where $\mathfrak{S}_{\lambda S} = \langle (S \mid \boxed{T}\,)$; T is of shape λ and standard on $\mathscr{P} \rangle_{\mathbb{K}}$ is the *Schur module* of shape λ associated to S. The symbol $(S \mid \boxed{T}\,)$ denotes the *right symmetrized* bitableau defined in [8]. To understand formula (17), the outer sum gives the (unique) decomposition of Super$^{[n]}$[$\mathscr{L}_2 \mid \mathscr{P}$] into isotypic components corresponding to shapes $\lambda \vdash 2n$; the inner sum provides a way to decompose each isotypic component into pl(V)-irreducibles, which are parametrized by standard tableaux S on $\mathscr{L}_2$ of shape λ; each set $\{(S \mid \boxed{T}); T$ is of shape λ and standard on $\mathscr{P}\}$ is a basis for the pl(V)-irreducible Schur module $\mathfrak{S}_{\lambda S}$. One can note that in the classical case $V = V_1$, the highest weight vector in $\mathfrak{S}_{\lambda S}$ is $(S \mid \mathrm{Der}_-(\lambda))$ with the highest weight $(\tilde{\lambda}_1, \tilde{\lambda}_2, ...)$, where

$$\mathrm{Der}_-(\lambda) = \begin{pmatrix} e_1^- & e_2^- & \cdots & e_{\tilde{\lambda}_1}^- \\ e_1^- & e_2^- & \cdots & e_{\tilde{\lambda}_2}^- \\ \cdots & & & \end{pmatrix}$$

is the *Deruyts tableau* of shape λ.

By the first Schur Lemma, every irreducible of $\mathrm{Super}^{[n]}[\mathscr{L}_2 \mid \mathscr{P}]$ is mapped under $\mathbf{U}$ either identically to $\mathbf{0}$ or isomorphically to an irreducible of $\mathbf{S}^n(\mathbf{S}^2(V))$. When λ is not even, by the proof of the regularization algorithm, it follows that $\mathbf{U} \cdot (S \mid \mathrm{Der}_-(\lambda)) = 0$, since all the terms in the second sum on the right side of (13) are identically zero for $T = \mathrm{Der}_-(\lambda)$. Hence all the isotypic components not corresponding to an even λ in (17) are in the kernel of $\mathbf{U}$. On the other hand, if $\lambda = (2p, 2q, ...)$ is even,

$$\begin{aligned} &\mathbf{U} \cdot (R_\lambda^+ \mid \mathrm{Der}_-(\lambda)) \\ &\quad = \mathbf{U} \cdot (\alpha_1 \alpha_1 \cdots \alpha_p \alpha_p \mid e_1 e_2 \cdots e_{2p}) \\ &\qquad \times \mathbf{U} \cdot (\alpha_{p+1} \alpha_{p+1} \cdots \alpha_{p+q} \alpha_{p+q} \mid e_1 e_2 \cdots e_{2q}) \cdots \\ &\quad = c \cdot \mathrm{Pfaffian}(M_{12 \cdots 2p}) \cdot \mathrm{Pfaffian}(M_{12 \cdots 2q}) \cdots \neq \mathbf{0}, \qquad c \in \mathbb{K}. \end{aligned}$$

Therefore each irreducible $\mathfrak{S}_{\lambda R_\lambda^+}$, λ even, is mapped under $\mathbf{U}$ isomorphically to an irreducible of $\mathbf{S}^n(\mathbf{S}^2(V))$. In particular the set

$$\bigcup_{\substack{\lambda \vdash 2n \\ \lambda \text{ even}}} \{\mathbf{U} \cdot (R_\lambda^+ \mid \boxed{T}); T \text{ is of shape } \lambda \text{ and standard on } \mathscr{P}\} \tag{18}$$

is linearly independent. Since $\mathbf{U}$ is surjective, the set in (18) is a basis of $\mathbf{S}^n(\mathbf{S}^2(V))$ according to the regularization algorithm. Of course, this set shares the same cardinality with the set

$$\bigcup_{\substack{\lambda \vdash 2n \\ \lambda \text{ even}}} \{\mathbf{U} \cdot (R_\lambda^+ \mid T); T \text{ is of shape } \lambda \text{ and standard on } \mathscr{P}\}.$$

Therefore the latter set is also a basis of $\mathbf{S}^n(\mathbf{S}^2(V))$, since it is a spanning set as we proved earlier. This proves the uniqueness part of formula (14) of Rota and Stein. To summarize, we have obtained the Gordan–Capelli series

$$
\begin{aligned}
\mathbf{S}^n(\mathbf{S}^2(V)) &= \bigoplus_{\substack{\lambda \vdash 2n \\ \lambda \text{ even}}} \mathbf{U} \cdot \mathfrak{S}_{\lambda R_\lambda^+} \\
&= \bigoplus_{\substack{\lambda \vdash 2n \\ \lambda \text{ even}}} \langle \mathbf{U} \cdot (R_\lambda^+ \mid \boxed{T}\,);\ T \text{ is standard on } \mathscr{P} \rangle_{\mathbb{K}}, \qquad (19)
\end{aligned}
$$

where the terms in the sum are *mutually* non-isomorphic irreducible $\mathrm{pl}(V)$-modules. This is a *multiplicity-free* decomposition of $\mathbf{S}^n(\mathbf{S}^2(V))$ into $\mathrm{pl}(V)$-irreducibles. Moreover, since

$$\mathfrak{S}_{\lambda R_\lambda^+} = \langle (R_\lambda^+ \mid \mathrm{Der}_-(\lambda)) \rangle_{\mathrm{pl}(V)},$$

whenever $\dim(V_1)$ is sufficiently large, we have

$$\mathbf{U} \cdot \mathfrak{S}_{\lambda R_\lambda^+} = \langle \mathrm{Pfaffian}(M_{12 \cdots \lambda_1}) \cdot \mathrm{Pfaffian}(M_{12 \cdots \lambda_2}) \cdots \rangle_{\mathrm{pl}(V)}$$

if $\lambda = (\lambda_1, \lambda_2, \ldots)$.

Next, we consider the $\mathbb{Z}_2$-companion symbolic operator $\bar{\mathbf{U}}$; we start from the complete decomposition

$$\mathrm{Super}^{[n]}[\bar{\mathscr{L}}_2 \mid \bar{\mathscr{P}}] = \bigoplus_{\lambda \vdash 2n} \bigoplus_{\substack{\mathbf{S} \text{ standard on } \bar{\mathscr{L}}_2 \\ \mathrm{sh}(\mathbf{S}) = \lambda}} \mathfrak{W}_{\lambda \mathbf{S}}, \qquad (20)$$

where

$$\mathfrak{W}_{\lambda \mathbf{S}} = \langle (\mathbf{S} \mid \boxed{\mathbf{T}}\,);\ \mathbf{T} \text{ is of shape } \lambda \text{ and standard on } \bar{\mathscr{P}} \rangle_{\mathbb{K}}$$

is called the *Weyl module* of shape λ associated to $\mathbf{S}$. Formula (20) is nothing but formula (17) for $\mathrm{Super}^{[n]}[\bar{\mathscr{L}}_2 \mid \bar{\mathscr{P}}]$. We can argue similarly that

$$
\begin{aligned}
\mathbf{S}^n(\mathbf{S}^2(V)) &= \bigoplus_{\substack{\lambda \vdash 2n \\ \tilde{\lambda} \text{ even}}} \bar{\mathbf{U}} \cdot \mathfrak{W}_{\lambda \mathbf{R}_\lambda^-} \\
&= \bigoplus_{\substack{\lambda \vdash 2n \\ \tilde{\lambda} \text{ even}}} \langle \bar{\mathbf{U}} \cdot (\mathbf{R}_\lambda^- \mid \boxed{\mathbf{T}}\,);\ \mathbf{T} \text{ is standard on } \bar{\mathscr{P}} \rangle_{\mathbb{K}} \qquad (21)
\end{aligned}
$$

which gives a Gordan–Capelli series for $\mathbf{S}^n(\mathbf{S}^2(V))$ different from the one given in (19). Again this is a *multiplicity-free* decomposition of $\mathbf{S}^n(\mathbf{S}^2(V))$ into $\mathrm{pl}(V)$-irreducibles. Let us compare (19) and (21). First of all, they should give the *same* (multiplicity-free) decomposition of $\mathbf{S}^n(\mathbf{S}^2(V))$ into $\mathrm{pl}(V)$-irreducibles; hence any irreducible $\mathbf{U} \cdot \mathfrak{S}_{\lambda R_\lambda^+}$ in (19) has to be $\mathrm{pl}(V)$-isomorphic to some irreducible $\bar{\mathbf{U}} \cdot \mathfrak{W}_{\mu \mathbf{R}_\mu^-}$ in (21). A natural candidate looks like $\mu = \tilde{\lambda}$. This is indeed the case. Consider the map

$$F_1 : \mathrm{Super}^{[n]}[\bar{\mathscr{L}}_2 \mid \bar{\mathscr{P}}] \to \mathrm{Super}^{[n]}[\mathscr{L}_2 \mid \mathscr{P}]$$

such that $F_1((\mathbf{y} \mid \mathbf{z})) = (y \mid z)$ for $\mathbf{y} \in \mathscr{L}_2$ and $\mathbf{z} \in \bar{\mathscr{P}}$. Clearly, the map F_1 is a pl(V)-isomorphism.

PROPOSITION 9. $F_1((\mathbf{R}_\lambda^- \mid \boxed{\mathbf{T}})) = c_\lambda(\boxed{R_{\tilde\lambda}^+} \mid \tilde{T})$ *for some non-zero* $c_\lambda \in \mathbb{K}$.

Proof. By definition,

$$\begin{aligned} F_1((\mathbf{R}_\lambda^- \mid \boxed{\mathbf{T}})) &= F_1(D(\mathbf{R}_\lambda^-)(\mathbf{Coder}_+(\lambda) \mid \mathbf{Der}_-(\lambda))(\mathbf{T})\mathfrak{a}) \\ &= c_\lambda D(R_{\tilde\lambda}^+)(\mathrm{Der}_-(\tilde\lambda) \mid \mathrm{Coder}_+(\tilde\lambda))(\tilde{T})\,\mathfrak{a} \\ &= c_\lambda(\boxed{R_{\tilde\lambda}^+} \mid \tilde{T}). \end{aligned}$$

In the above formulas we have used the notation and the facts in [9, 11]. ∎

Therefore we have the pl(V)-isomorphism

$$F_1 : \mathfrak{W}_{\lambda\mathbf{R}_\lambda^-} \to \langle(\boxed{R_{\tilde\lambda}^+} \mid T);\, T \text{ is standard on } \mathscr{P}\rangle_{\mathbb{K}}$$

between pl(V)-irreducibles. Moreover

$$\begin{aligned} &\langle(\boxed{R_{\lambda}^+} \mid T);\, T \text{ is standard on } \mathscr{P}\rangle_{\mathbb{K}} \\ &\quad = \langle(\boxed{R_{\lambda}^+} \mid \mathrm{Der}_-(\lambda))\rangle_{\mathrm{pl}(V)} \\ &\quad = \langle(\boxed{R_{\lambda}^+} \mid \boxed{T});\, T \text{ is standard on } \mathscr{P}\rangle_{\mathbb{K}}. \end{aligned}$$

PROPOSITION 10. *The map*

$$F_2 : \mathfrak{S}_{\lambda R_\lambda^+} \to \langle(\boxed{R_{\tilde\lambda}^+} \mid T);\, T \textit{ is standard on } \mathscr{P}\rangle_{\mathbb{K}},$$

such that $F_2((R_{\tilde\lambda}^+ \mid \boxed{T})) = (\boxed{R_{\tilde\lambda}^+} \mid \boxed{T})$, *is a* pl($V$)-*isomorphism.*

Proof. The assertion follows from Proposition 8 of [8]. ∎

Combining F_1 and F_2 we obtain the pl(V)-isomorphism

$$F_2^{-1} \circ F_1 : \mathfrak{W}_{\lambda\mathbf{R}_\lambda^-} \to \mathfrak{S}_{\tilde\lambda R_{\tilde\lambda}^+}$$

and hence the pl(V)-isomorphism

$$\mathbf{U} \circ F_2^{-1} \circ F_1 \circ \bar{\mathbf{U}}^{-1} : \bar{\mathbf{U}} \cdot \mathfrak{W}_{\lambda\mathbf{R}_\lambda^-} \to \mathbf{U} \cdot \mathfrak{S}_{\tilde\lambda R_{\tilde\lambda}^+},$$

where $\bar{\mathbf{U}}^{-1}$ is the inverse of $\bar{\mathbf{U}}$ when restricted on $\bar{\mathbf{U}} \cdot \mathfrak{W}_{\lambda\mathbf{R}_\lambda^-}$. This is the desired 1–1 correspondence between the pl(V)-irreducibles in (21) and the ones in (19). Note that the element $\bar{\mathbf{U}} \cdot (\mathbf{R}_{\lambda}^{-} \mid \boxed{\mathbf{T}})$ is not mapped to $\mathbf{U} \cdot (R_{\tilde{\lambda}}^{+} \mid \boxed{T})$ under the isomorphism, but to a rather complicated expression. The sets

$$\{\bar{\mathbf{U}} \cdot (\mathbf{R}_{\lambda}^{-} \mid \boxed{\mathbf{T}}); \mathbf{T} \text{ is of shape } \lambda \text{ and standard on } \bar{\mathscr{P}}\},$$

and

$$\{\mathbf{U} \cdot (R_{\tilde{\lambda}}^{+} \mid \boxed{T}); T \text{ is of shape } \tilde{\lambda} \text{ and standard on } \mathscr{P}\}$$

provide two essentially different bases for the same $\mathrm{pl}(V)$-irreducible of $\mathbf{S}^n(\mathbf{S}^2(V))$.

5. Applications: E. Pascal Theorems for Orthogonal and Symplectic Invariants

As applications of results in the previous sections, we derive the classical E. Pascal theorems for inner products and for symplectic products. They are usually regarded as the second fundamental theorems for orthogonal invariants and symplectic invariants [57].

Let us start with the inner products. Let $V = V_0$, $\dim(V) = d$. The set $\{e_1^+, e_2^+, ..., e_d^+\}$ is a basis of V. Fix an integer $m \geqslant 1$ and denote by $\mathbb{K}[e_{ik}]$ the polynomial algebra generated by the free variables e_{ik}, $i = 1, 2, ..., d$ and $k = 1, 2, ..., m$. For every pair (i, j), $i, j = 1, 2, ..., d$, set $(e_i, e_j) = \sum_{k=1}^{m} e_{ik} e_{jk} \in \mathbb{K}[e_{ik}]$. Following the notation in [17], we denote by $\bar{R}_m = \mathbb{K}[(e_i, e_j)]$ the subalgebra of $\mathbb{K}[e_{ik}]$ generated by the "inner products" (e_i, e_j), $i, j = 1, 2, ..., d$. Since $V = V_0$, we have $\mathrm{pl}(V) = \mathrm{gl}(V)$, the classical general linear Lie algebra of V. The polynomial algebra $\mathbb{K}[e_{ik}]$ is a $\mathrm{gl}(V)$-module by setting $E_{ij} \cdot e_{hk} = \delta_{jh} e_{ik}$ and extending it by (even) derivation. Since

$$E_{ij} \cdot (e_k, e_h) = (E_{ij} \cdot e_k, e_h) + (e_k, E_{ij} \cdot e_h),$$

the algebra $\bar{R}_m$ is a $\mathrm{gl}(V)$-submodule of $\mathbb{K}[e_{ik}]$. For the sake of simplicity, we consider the vector space

$$\bar{R}_m^n \stackrel{\text{def}}{=} \bar{R}_m \cap \mathbb{K}^{2n}[e_{ik}],$$

where $\mathbb{K}^{2n}[e_{ik}]$ is the $\mathbb{Z}$-homogeneous component of degree $2n$ of $\mathbb{K}[e_{ik}]$. One can regard $\bar{R}_m^n$ as the subspace of $\mathbb{K}[e_{ik}]$ of all homogeneous polynomials of degree n in the "inner products" (e_i, e_j). It is clear that $\bar{R}_m^n$ is also a $\mathrm{gl}(V)$-submodule of $\mathbb{K}[e_{ik}]$. The map $[e_i^+ e_j^+] \to (e_i, e_j)$ uniquely extends to a $\mathrm{gl}(V)$-equivariant algebraic epimorphism

$$\Phi_m : \mathrm{Sym}(\mathrm{Sym}^2(V)) \to \bar{R}_m.$$

The restriction of Φ_m on $\mathrm{Sym}^n(\mathrm{Sym}^2(V))$ will be denoted by Φ_m^n. We recall that $\mathrm{Sym}^n(\mathrm{Sym}^2(V))$ has a basis given by the set

$$\{\bar{\mathbf{U}} \cdot (\mathbf{R}_\lambda^- \mid \boxed{\mathrm{T}}); \mathbf{T} \text{ is standard on } \bar{\mathscr{P}}, \tilde{\lambda} \vdash 2n \text{ is even}, \lambda_1 \leqslant d\},$$

where the last condition $\lambda_1 \leqslant d$ is implied by the hook condition $\lambda_{d_1+1} \leqslant d_0$.

PROPOSITION 11. *The kernel of the map Φ_m^n is given by*

$$\ker(\Phi_m^n) = \langle \bar{\mathbf{U}} \cdot (\mathbf{R}_\lambda^- \mid \boxed{\mathrm{T}}); \mathbf{T} \textit{ is standard on } \bar{\mathscr{P}}, \tilde{\lambda} \vdash 2n \textit{ is even}, \lambda_1 > m \rangle_{\mathbb{K}}.$$

Proof. By the first Schur Lemma, the $\mathrm{gl}(V)$-irreducible $\bar{\mathbf{U}} \cdot \mathfrak{W}_{\lambda \mathbf{R}_\lambda^-}$ is in the kernel of Φ_m^n if and only if

$$\Phi_m^n(\bar{\mathbf{U}} \cdot (\mathbf{R}_\lambda^- \mid \mathbf{Der}_-(\lambda))) = 0.$$

Let $\lambda = (p, p, q, q, ...)$; we have

$$\begin{aligned}
&\Phi_m^n(\bar{\mathbf{U}} \cdot (\mathbf{R}_\lambda^- \mid \mathbf{Der}_-(\lambda))) \\
&\quad = \pm \Phi_m^n\left(\bar{\mathbf{U}} \cdot \begin{pmatrix} \boldsymbol{\alpha}_1 \cdots \boldsymbol{\alpha}_p & \mathbf{e}_1 \cdots \mathbf{e}_p \\ \boldsymbol{\alpha}_1 \cdots \boldsymbol{\alpha}_p & \mathbf{e}_1 \cdots \mathbf{e}_p \end{pmatrix} \bar{\mathbf{U}} \cdot \begin{pmatrix} \boldsymbol{\alpha}_{p+1} \cdots \boldsymbol{\alpha}_{p+q} & \mathbf{e}_1 \cdots \mathbf{e}_q \\ \boldsymbol{\alpha}_{p+1} \cdots \boldsymbol{\alpha}_{p+q} & \mathbf{e}_1 \cdots \mathbf{e}_q \end{pmatrix} \cdots \right) \\
&\quad = \pm \Phi_m^n(p! \det([e_i e_j]_{1 \leqslant i,j \leqslant p}) \cdot q! \det([e_i e_j]_{1 \leqslant i,j \leqslant q}) \cdots) \\
&\quad = \pm p! \det((e_i, e_j)_{1 \leqslant i,j \leqslant p}) \cdot q! \det((e_i, e_j)_{1 \leqslant i,j \leqslant q}) \cdots. \qquad (22)
\end{aligned}$$

Now consider the matrix $(e_i, e_j)_{1 \leqslant i,j \leqslant p}$; let (e_{ik}) be the $p \times m$ matrix with its ith row being $(e_{i1}, e_{i2}, ..., e_{im})$. We have

$$((e_i, e_j)_{1 \leqslant i,j \leqslant p}) = (e_{ik}) \cdot (e_{ik})^{\mathrm{T}}.$$

Therefore, the rank of the $p \times p$ matrix $((e_i, e_j)_{1 \leqslant i,j \leqslant p})$ is $\min(p, m)$. Hence

$$\det((e_i, e_j)_{1 \leqslant i,j \leqslant p}) = 0$$

if and only if $p > m$; thus (22) implies that

$$\Phi_m^n(\bar{\mathbf{U}} \cdot (\mathbf{R}_\lambda^- \mid \mathbf{Der}_-(\lambda))) = 0 \qquad \text{if and only if} \quad \lambda_1 = p > m.$$

Since $\mathrm{Sym}(\mathrm{Sym}^2(V))$ is a multiplicity-free $\mathrm{gl}(V)$-module, the proposition follows. ∎

As a corollary, we have

PROPOSITION 12 (E. Pascal Theorem for Inner Products). *The kernel of the map Φ_m^n is given by*

$$\ker(\Phi_m^n) = \langle \bar{\mathbf{U}} \cdot (\mathbf{R}_\lambda^- \mid \mathbf{T}); \mathbf{T} \textit{ is standard on } \bar{\mathscr{P}}, \lambda \vdash 2n \textit{ is even}, \lambda_1 > m \rangle_{\mathbb{K}}.$$

Proof. This proposition immediately follows from the property that, for a given shape $\lambda \vdash 2n$,

$$\langle (\mathbf{S} \mid \boxed{\mathbf{T}}); \mathbf{S} \text{ and } \mathbf{T} \text{ are standard on } \bar{\mathscr{L}} \text{ and } \bar{\mathscr{P}}, \operatorname{sh}(\mathbf{S}) = \operatorname{sh}(\mathbf{T}) \geqslant \lambda \rangle_{\mathbb{K}}$$

$$= \langle (\mathbf{S} \mid \mathbf{T}); \mathbf{S} \text{ and } \mathbf{T} \text{ are standard on } \bar{\mathscr{L}} \text{ and } \bar{\mathscr{P}}, \operatorname{sh}(\mathbf{S}) = \operatorname{sh}(\mathbf{T}) \geqslant \lambda \rangle_{\mathbb{K}},$$

see [8, 9]. ▮

From Proposition 12, one can infer the traditional formulation of the second fundamental theorem for orthogonal invariants (Weyl [57]), that is:

PROPOSITION 13. *The ideal of the relations among the inner products* (e_i, e_j), $i, j = 1, 2, ..., d$, *where* $d > m$, *is generated by the* $(m+1) \times (m+1)$ *minors of the symmetric matrix* $([e_i e_j]_{1 \leqslant i, j \leqslant d})$.

Proof. From the previous proposition, the ideal is generated by

$$\bar{\mathbf{U}} \cdot \begin{pmatrix} \boldsymbol{\alpha}_1 \cdots \boldsymbol{\alpha}_p & \mathbf{e}_{i_1} \cdots \mathbf{e}_{i_p} \\ \boldsymbol{\alpha}_1 \cdots \boldsymbol{\alpha}_p & \mathbf{e}_{j_1} \cdots \mathbf{e}_{j_p} \end{pmatrix} = (-1)^p p! \det([e_{i_s} e_{j_t}]_{1 \leqslant s, t \leqslant p}), \qquad p > m.$$

The proposition follows from the Laplace expansion of $\det([e_{i_s} e_{j_t}]_{1 \leqslant s, t \leqslant p})$ with respect to minors of size $m+1$. ▮

We proceed to discuss the E. Pascal Theorem for symplectic invariants. Let $V = V_1$, $\dim(V) = d$. The set $\{e_1^-, e_2^-, ..., e_d^-\}$ is a basis of V. Fix an even integer $m = 2h \geqslant 2$; the polynomial algebra $\mathbb{K}[e_{ik}]$ is defined as earlier. Let A be the $m \times m$ matrix

$$\begin{pmatrix} 0 & I_h \\ -I_h & 0 \end{pmatrix}.$$

For every pair (i, j), $i, j = 1, 2, ..., d$, set

$$\langle e_i, e_j \rangle = (e_{i1}, e_{i2}, ..., e_{im}) A (e_{i1}, e_{i2}, ..., e_{im})^{\mathrm{T}}$$

$$= \sum_{k=1}^{h} (-e_{i\,h+k} e_{jk} + e_{ik} e_{j\,h+k}).$$

We denote by R_m the subalgebra of $\mathbb{K}[e_{ik}]$ generated by the "symplectic products" $\langle e_i, e_j \rangle$, $i, j = 1, 2, ..., d$, and by R_m^n the subspace of R_m of all homogeneous polynomials of degree n in the "symplectic products" $\langle e_i, e_j \rangle$. We still have $E_{ij} \cdot \langle e_k, e_h \rangle = \langle E_{ij} \cdot e_k, e_h \rangle + \langle e_k, E_{ij} \cdot e_h \rangle$, so both R_m and R_m^n are $\mathrm{gl}(V)$-submodules of $\mathbb{K}[e_{ik}]$. The map $[e_i^- e_j^-] \to \langle e_i, e_j \rangle$ uniquely extends to a $\mathrm{gl}(V)$-equivariant algebraic epimorphism

$$\Psi_m \colon \operatorname{Sym}(\operatorname{Sym}^2(V)) \to R_m.$$

We denote its restriction on $\mathrm{Sym}^n(\Lambda^2(V))$ by Ψ^n_m. Recall that $\mathrm{Sym}^n(\Lambda^2(V))$ has a basis given by the set

$$\{\mathrm{U}\cdot(R^+_\lambda \mid \boxed{T}\,);\ T \text{ is standard on } \mathscr{P},\ \lambda \vdash 2n \text{ is even},\ \lambda_1 \leqslant d\}.$$

As an analog of Proposition 12, we have

PROPOSITION 14. *The kernel of the map Ψ^n_m is given by*

$$\ker(\Psi^n_m) = \langle \mathrm{U}\cdot(R^+_\lambda \mid \boxed{T}\,);\ T \textit{ is standard on } \mathscr{P},\ \lambda \vdash 2n \textit{ is even},\ \lambda_1 > m\rangle_{\mathbb{K}}.$$

Proof. The proof is similar to the one for Proposition 11, except that now $\mathrm{U}\cdot(R^+_\lambda \mid \boxed{T}\,)$ is a product of factors of the type $\mathrm{Pfaffian}([e_ie_j]_{1\leqslant i,j\leqslant 2p})$, up to a scalar. Note that

$$\begin{aligned}[]
&[\Psi^p_m(\mathrm{Pfaffian}([e_ie_j]_{1\leqslant i,j\leqslant 2p}))]^2\\
&\quad= \Psi^{2p}_m(\mathrm{Pfaffian}^2([e_ie_j]_{1\leqslant i,j\leqslant 2p}))\\
&\quad= \Psi^{2p}_m(\det([e_ie_j]_{1\leqslant i,j\leqslant 2p})) = \det(\langle e_i, e_j\rangle_{1\leqslant i,j\leqslant 2p}).
\end{aligned}$$

Hence

$$[\Psi^p_m(\mathrm{Pfaffian}([e_ie_j]_{1\leqslant i,j\leqslant 2p}))]^2 = 0 \Leftrightarrow \det(\langle e_i, e_j\rangle_{1\leqslant i,j\leqslant 2p}) = 0.$$

On the other hand, let (e_{ik}) be the $2p\times m$ matrix with its ith row being $(e_{i1}, e_{i2}, ..., e_{im})$; we have

$$(\langle e_i, e_j\rangle_{1\leqslant i,j\leqslant 2p}) = (e_{ik})\,A(e_{ik})^{\mathrm{T}}.$$

Hence the rank of the matrix $(\langle e_i, e_j\rangle_{1\leqslant i,j\leqslant 2p})$ is $\min(m, 2p)$; therefore

$$\det(\langle e_i, e_j\rangle_{1\leqslant i,j\leqslant 2p}) = 0 \Leftrightarrow 2p > m.$$

We conclude that

$$\Psi^n_m(\mathrm{U}\cdot(R^+_\lambda \mid \mathrm{Der}_-(\lambda))) = 0 \Leftrightarrow \lambda_1 > m.$$

Since $\mathrm{Sym}(\Lambda^2(V))$ is a multiplicity-free $\mathrm{gl}(V)$-module, the proposition follows. ∎

Proposition 14 implies the following two propositions, which are analogous to Proposition 12 and 13.

PROPOSITION 15 (E. Pascal Theorem for Symplectic Products). *The kernel of the map Ψ^n_m is given by*

$$\ker(\Psi^n_m) = \langle \mathrm{U}\cdot(R^+_\lambda \mid T);\ T \textit{ is standard on } \mathscr{P},\ \lambda \vdash 2n \textit{ is even},\ \lambda_1 > m\rangle_{\mathbb{K}}.$$

PROPOSITION 16. *The ideal of the relations among the symplectic products* $\langle e_i, e_j \rangle$, $i, j = 1, 2, ..., d$, *where* $d \geqslant m+2$, *is generated by the Pfaffians of the* $(m+2)\times(m+2)$ *submatrices of rows* $i_1, i_2, ..., i_{m+2}$ *and columns* $i_1, i_2, ..., i_{m+2}$ *of the skew-symmetric matrix* $(\langle e_i, e_j \rangle_{1 \leqslant i,j \leqslant d})$.

6. GORDAN–CAPELLI SERIES AND STRAIGHTENING FORMULAS FOR $\Lambda^n(\mathbf{S}^2(V))$

Along the lines discussed in Sections 3 and 4, we describe two different straightening formulas and two different Gordan–Capelli series for the plethystic algebra $\Lambda^n(\mathbf{S}^2(V))$. The result about the explicit complete decomposition of $\Lambda^n(\mathbf{S}^2(V))$ into irreducibles is new, although the abstract structures of $\Lambda^n(\mathrm{Sym}^2(V_0))$ and $\Lambda^n(\Lambda^2(V_1))$ are known from the theory of symmetric functions. To be precise, we recall that the coefficients c_λ and d_λ in the expansions

$$e_n \circ h_2 = \sum_\lambda c_\lambda s_\lambda, \qquad e_n \circ e_2 = \sum_\lambda d_\lambda s_\lambda$$

of the plethystic compositions $e_n \circ h_2$, $e_n \circ e_2$ of symmetric functions into linear combinations of Schur functions are well known and they correspond to the multiplicities of the classical Schur modules in $\Lambda^n(\mathrm{Sym}^2(V_0))$ and in $\Lambda^n(\Lambda^2(V_1))$ with respect to shape λ [40].

We defined earlier the symbolic operator

$$\mathbf{U}'\colon \mathrm{Super}^{[n]}[\mathscr{L}_2 \mid \mathscr{P}_{x^-}] \to \Lambda^n(\mathbf{S}^2(V)).$$

Note that the map $F\colon \mathrm{Super}^{[n]}[\mathscr{L}_2 \mid \mathscr{P}_{x^-}] \to \mathrm{Super}^{[n]}[\mathscr{L}_2 \mid \mathscr{P}]$, given by

$$F((\alpha_1 \mid x)(\alpha_2 \mid x) \cdots (\alpha_n \mid x)\, \mathbf{m}) = \mathbf{m},$$

is a $\mathrm{pl}(V)$-isomorphism. So the Gordan–Capelli series of $\mathrm{Super}^{[n]}[\mathscr{L}_2 \mid \mathscr{P}_{x^-}]$ is given by

$$\mathrm{Super}^{[n]}[\mathscr{L}_2 \mid \mathscr{P}_{x^-}] = \bigoplus_{\lambda \vdash 2n} \bigoplus_{\substack{S \text{ standard on } \mathscr{L}_2 \\ \mathrm{sh}(S)=\lambda}} (\alpha_1 \mid x)(\alpha_2 \mid x) \cdots (\alpha_n \mid x)\, \mathfrak{S}_{\lambda S}. \tag{23}$$

We are interested in finding the condition for λ such that

$$\mathbf{U}'((\alpha_1 \mid x)(\alpha_2 \mid x) \cdots (\alpha_n \mid x)(R \mid \mathrm{Der}_-(\lambda))) \neq 0,$$

for some tableau R of shape λ. A tableau R on $\mathscr{L}_2$ is said to be *permissible* whenever for any pair α_i and α_j: (i) no row of R contains both two α_i and two α_j; (ii) if one of α_i and one of α_j appear in the same row of R, then the other α_i and the other α_j appear in different rows of R. For example, both the following tableaux are not permissible:

$$\begin{pmatrix} \alpha_1 & \alpha_1 & \alpha_2 & \alpha_2 \\ \alpha_3 & \alpha_3 & & \end{pmatrix}, \quad \begin{pmatrix} \alpha_1 & \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_2 & \alpha_3 & & \end{pmatrix}.$$

PROPOSITION 17. *If R is not permissible on $\mathscr{L}_2$, then*

$$\mathbf{U}'((\alpha_1 \mid x)(\alpha_2 \mid x) \cdots (\alpha_n \mid x)(R \mid T)) = 0.$$

Proof. Write $\mathbf{m} = (\alpha_1 \mid x)(\alpha_2 \mid x) \cdots (\alpha_n \mid x)(R \mid T)$. Let α_i and α_j be a violation pair in R and $\sigma = (ij) \in S_n$. Then

$$\sigma \cdot (\alpha_1^+ \mid x^-)(\alpha_2^+ \mid x^-) \cdots (\alpha_n^+ \mid x^-) = -(\alpha_1 \mid x)(\alpha_2 \mid x) \cdots (\alpha_n \mid x),$$

and $\sigma \cdot (R \mid T) = (R \mid T)$. So $\sigma(\mathbf{m}) = -\mathbf{m}$ and $\mathbf{U}' \cdot \sigma(\mathbf{m}) = -\mathbf{U}' \cdot \mathbf{m}$. On the other hand, $\mathbf{U}' \cdot \sigma(\mathbf{m}) = \mathbf{U}' \cdot \mathbf{m}$ by Proposition 7; therefore

$$\mathbf{U}' \cdot \mathbf{m} = 0,$$

as desired. ∎

Recall that the *Frobenius notation* $(a_1, a_2, ..., a_p;\ b_1, b_2, ..., b_q)$, where $a_i > a_{i+1}$ and $b_i > b_{i+1}$, for a shape λ is defined such that a_i is the number of cells in the ith column below the main diagonal in the Ferrers diagram of λ and b_j is the number of cells in the jth row on the right of the main diagonal. For example,

$$(3, 1; 4, 2, 1) = \begin{array}{ccccc} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & & & & \end{array}$$

LEMMA 18. *If there is a permissible tableau R on $\mathscr{L}_2$ of shape $\lambda = (a_1, a_2, ..., a_p;\ b_1, b_2, ..., b_q)$, then $a_1 \geqslant b_1 - 1$.*

Proof. Let R be such a permissible tableau. Then the first row of R has to be either $\alpha_1\alpha_1\alpha_2\alpha_3 \cdots \alpha_{b_1}$ or $\alpha_1\alpha_2\alpha_3 \cdots \alpha_{b_1+1}$ (up to changing the indices of α_i). In both cases, since R is permissible, the other $\alpha_2, \alpha_3, ..., \alpha_{b_1}$ have to appear in $b_1 - 1$ different rows. Therefore the number of rows, which is $a_1 + 1$, has to be at least b_1. ∎

Furthermore, in order to have

$$\mathbf{U}'((\alpha_1 \mid x)(\alpha_2 \mid x) \cdots (\alpha_n \mid x)(R \mid \mathrm{Der}_-(\lambda))) \neq 0,$$

it is clear that one can find at least one non-zero monomial $[e_{i_1}^- e_{i_2}^-]$ $[e_{j_1}^- e_{j_2}^-] \cdots [e_{h_1}^- e_{h_2}^-]$ in $\Lambda^n(\mathbf{S}^2(V))$, such that its content is the same as the content of $\mathrm{Der}_-(\lambda)$. Note that such a monomial is non-zero if and only if the two-element sets $\{e_{i_1}, e_{i_2}\}$, $\{e_{j_1}, e_{j_2}\}$, ..., $\{e_{h_1}, e_{h_2}\}$ are pairwise distinct and the two elements of each set are different, since in $\Lambda^n(\mathbf{S}^2(V))$

$$[e_i^- e_j^-] = -[e_j^- e_i^-] \qquad \text{and} \qquad [e_i^- e_j^-][e_h^- e_k^-] = -[e_h^- e_k^-][e_i^- e_j^-].$$

Such a collection of sets $\{e_{i_1}, e_{i_2}\}$, $\{e_{j_1}, e_{j_2}\}$, ..., $\{e_{h_1}, e_{h_2}\}$ is called a *permissible splitting* of $\mathrm{Der}_-(\lambda)$.

LEMMA 19. *Let* $\lambda = (a_1, a_2, ..., a_p;\ b_1, b_2, ..., b_q)$. *If* $\mathrm{Der}_-(\lambda)$ *admits a permissible splitting, then* $a_1 \leqslant b_1 - 1$.

Proof. Let $\{e_1, e_{i_1}\}$, $\{e_1, e_{i_2}\}$, ..., $\{e_1, e_{i_{a_1+1}}\}$ be the pairs containing e_1 in such a spitting. Then $e_1, e_{i_1}, e_{i_2}, ..., e_{i_{a_1+1}}$ must be $a_1 + 2$ distinct symbols. Note that $b_1 + 1$ is the number of different symbols in $\mathrm{Der}_-(\lambda)$, so the assertion follows. ∎

Combining Lemma 18 and Lemma 19, we infer that if

$$\mathbf{U}'((\alpha_1 \mid x)(\alpha_2 \mid x) \cdots (\alpha_n \mid x)(R \mid \mathrm{Der}_-(\lambda))) \neq 0,$$

then $a_1 = b_1 - 1$. Iterating the process, we have:

THEOREM 4. *If* $\mathbf{U}'((\alpha_1 \mid x)(\alpha_2 \mid x) \cdots (\alpha_n \mid x)(R \mid \mathrm{Der}_-(\lambda))) \neq 0$ *for some tableau* R *on* $\mathscr{L}_2$, *then* $\lambda = ((b_1 - 1), (b_2 - 1), ..., (b_p - 1);\ b_1, b_2, ..., b_p)$ *for some* $b_1, b_2, ..., b_p$ *in Frobenius notation.*

Proof. Let $\lambda = (a_1, a_2, ..., a_p;\ b_1, b_2, ..., b_q)$. By previous arguments, we know that $a_1 = b_1 - 1$. If the first row of R is $\alpha_1 \alpha_2 \cdots \alpha_{b_1+1}$ (up to reindexing), then the other $\alpha_1, \alpha_2, ..., \alpha_{b_1+1}$ have to appear in $b_1 + 1$ rows different from the first row. So the number of rows equals $a_1 + 1 \geqslant b_1 + 2$, which is a contradiction. Hence the first row of R has to be $\alpha_1 \alpha_1 \alpha_2 \cdots \alpha_{b_1}$ (up to reindexing). In this case, every other row contains one and only one letter from the set $\{\alpha_2, \alpha_3, ..., \alpha_{b_1}\}$. Say the ith row contains α_i, $2 \leqslant i \leqslant b_1$. Sorting the rows of R, we can assume that α_i is the first letter on the ith row, so that R has the following layout:

$$R = \begin{pmatrix} \alpha_1 & \alpha_1 & \alpha_2 & \alpha_3 \cdots \alpha_{b_1} \\ \alpha_2 & & & \\ \alpha_3 & & R' & \\ \cdots & & & \\ \alpha_{b_1} & & & \end{pmatrix}.$$

Now the first row of R' has to be of the form $\alpha_{b_1+1}\alpha_{b_1+2}\cdots\alpha_{b_1+b_2+1}$ or $\alpha_{b_1+1}\alpha_{b_1+1}\alpha_{b_1+2}\cdots\alpha_{b_1+b_2}$. Reasoning similarly as in proving Lemma 18, we have $a_2 \geqslant b_2 - 1$. On the other hand, considering a permissible splitting of $\mathrm{Der}_-(\lambda)$, we know that the pairs containing e_1 are given by $\{e_1, e_2\}$, $\{e_1, e_3\}, ..., \{e_1, e_{b_1+1}\}$ according to the proof of Lemma 19. Now let $\{e_1, e_2\}$ and $\{e_2, e_{i_1}\}$, $\{e_2, e_{i_2}\}, ..., \{e_2, e_{i_{a_2+1}}\}$ be the pairs containing e_2, then $a_2 \leqslant b_2 - 1$ by an argument similar to the proof of Lemma 19. Hence $a_2 = b_2 - 1$; iterating we have $a_i = b_i - 1$ for $i = 1, 2, ..., p$ and $p = q$. ▮

Partitions of the form $\lambda = ((b_1 - 1), (b_2 - 1), ..., (b_p - 1); b_1, b_2, ..., b_p)$ will be called *partitions of Frobenius type*. Given such a partition, let F_λ^+ be the standard tableau on L such that its ith rim is of the form

$$\begin{array}{lllll}
a_{s_{i-1}+1} & a_{s_{i-1}+1} & a_{s_{i-1}+2} & a_{s_{i-1}+3} & \cdots a_{s_i} \\
a_{s_{i-1}+2} & & & & \\
a_{s_{i-1}+3} & & & & \\
\cdots & & & & \\
a_{s_i} & & & &
\end{array}$$

where $s_i = b_1 + b_2 + \cdots + b_i$. For example

$$F^+_{(4,4,2)} = \begin{pmatrix} \alpha_1 & \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_2 & \alpha_4 & \alpha_4 & \alpha_5 \\ \alpha_3 & \alpha_5 & & \end{pmatrix}.$$

The following proposition can be drawn from the previous proof.

PROPOSITION 20 (Regularization Algorithm for $\Lambda^n(\mathbf{S}^2(V))$). *If* $\mathbf{U}'((\alpha_1 \mid x)(\alpha_2 \mid x)\cdots(\alpha_n \mid x)(R \mid \mathrm{Der}_-(\lambda))) \neq 0$, *then* λ *has to be of Frobenius type* $((b_1 - 1), (b_2 - 1), ..., (b_p - 1); b_1, b_2, ..., b_p)$ *and* $\pi \cdot R = \sigma \cdot F_\lambda^+$ *for some row stabilizer* π *of* R *and some* $\sigma \in S_n$. *The same statement is true for* $\mathbf{U}'((\alpha_1 \mid x)(\alpha_2 \mid x)\cdots(\alpha_n \mid x)(R \mid \boxed{T}))$. *Furthermore, we have*

$$\begin{aligned}
&\mathbf{U}'((\alpha_1 \mid x)(\alpha_2 \mid x)\cdots(\alpha_n \mid x)(R \mid \mathrm{Der}_-(\lambda))) \\
&\quad = c[e_1e_2][e_1e_3]\cdots[e_1e_{\lambda_1}][e_2e_3][e_2e_4]\cdots[e_2e_{\lambda_2}]\cdots \\
&\quad\quad \cdots[e_pe_{p+1}][e_pe_{p+2}]\cdots[e_pe_{\lambda_p}],
\end{aligned} \tag{24}$$

for some integer coefficient c.

Proof. To see (24), one just notes that the pairs $[e_ie_j]$ on the right side are the only permissible splitting of $\mathrm{Der}_-(\lambda)$. ▮

The expression on the right side of (24) (without the scalar c) will be denoted by $\mathscr{F}_\lambda^+$.

PROPOSITION 21. *In formula* (24), *c is a non-zero coefficient.*

We present two proofs.

Proof 1. Let

$$\mathrm{Coder}_+(\lambda) = \begin{pmatrix} e_1' & e_1' & e_1' & \cdots \\ e_2' & e_2' & \cdots & \\ e_3' & \cdots & & \\ \cdots & & & \end{pmatrix}$$

be a standard tableau of shape λ, were each e_i' is positive. It is clear that

$$\mathbf{U}'((\alpha_1 \mid x)(\alpha_2 \mid x) \cdots (\alpha_n \mid x)(F_\lambda^+ \mid \mathrm{Coder}_+(\lambda))) = k \prod [e_i' e_j'] \neq 0, \qquad (25)$$

where k is a non-zero integer, since any pair $[e_i' e_j']$ appears at most once in the product. For example

$$(\alpha_1 \mid x)(\alpha_2 \mid x) \cdots (\alpha_5 \mid x) \left(\begin{array}{cccc|cccc} \alpha_1 & \alpha_1 & \alpha_2 & \alpha_3 & e_1' & e_1' & e_1' & e_1' \\ \alpha_2 & \alpha_4 & \alpha_4 & \alpha_5 & e_2' & e_2' & e_2' & e_2' \\ \alpha_3 & \alpha_5 & & & e_3' & e_3' & & \end{array} \right)$$

$$= 4!^2\, 2!\, (\alpha_1 \mid x) \cdots (\alpha_5 \mid x)(\alpha_1 \mid e_1')(\alpha_1 \mid e_1')(\alpha_2 \mid e_1')(\alpha_3 \mid e_1')(\alpha_2 \mid e_2')$$
$$\times (\alpha_4 \mid e_2')(\alpha_4 \mid e_2')(\alpha_5 \mid e_2')(\alpha_3 \mid e_3')(\alpha_5 \mid e_3') \xrightarrow{\mathbf{U}'} 4!^2\, 2!\, [e_1' e_1']$$
$$\times [e_1' e_2'][e_1' e_3'][e_2' e_2'][e_2' e_3'].$$

Straighten $(F_\lambda^+ \mid \mathrm{Coder}_+(\lambda))$ by right-symmetrized bitableaux,

$$(F_\lambda^+ \mid \mathrm{Coder}_+(\lambda)) = \sum_{\substack{\mathrm{sh}(T) = \lambda \\ T \text{ standard}}} c_T (F_\lambda^+ \mid \boxed{T}) + \sum_{\substack{\mathrm{sh}(S \mid T) > \lambda \\ (S \mid T) \text{ standard}}} d_{ST} (S \mid \boxed{T}).$$

Note that if $\mathrm{sh}(S \mid T) > \lambda$ in dominance order, then $\mathrm{sh}(S \mid T)$ cannot be of Frobenius type. Hence the second sum above is mapped to zero by $\mathbf{U}'$ and so

$$\mathbf{U}'((\alpha_1 \mid x)(\alpha_2 \mid x) \cdots (\alpha_n \mid x)(F_\lambda^+ \mid \mathrm{Coder}_+(\lambda)))$$
$$= \mathbf{U}'((\alpha_1 \mid x)(\alpha_2 \mid x) \cdots (\alpha_n \mid x) \sum_{\substack{\mathrm{sh}(T) = \lambda \\ T \text{ standard}}} c_T (F_\lambda^+ \mid \boxed{T})),$$

which is non-vanishing by (25). Therefore when restricted on the irreducible $(\alpha_1 \mid x)(\alpha_2 \mid x) \cdots (\alpha_n \mid x)\, \mathfrak{S}_{\lambda F_\lambda^+}$, the symbolic operator $\mathbf{U}'$ is a $\mathrm{pl}(V)$-isomorphism by the first Schur Lemma and so $c \neq 0$ in (24).

Proof 2. We prove the assertion by contradiction. If $c=0$, then by Proposition 20,

$$\mathbf{U}' \cdot ((\alpha_1 \mid x)(\alpha_2 \mid x) \cdots (\alpha_n \mid x)(R \mid \boxed{T}\,)) = 0$$

for all $(R \mid \boxed{T}\,)$ of shape λ. Knowing that $\mathbf{U}'$ is surjective, let $\mathbf{m}$ be a preimage of $\mathscr{F}_\lambda^+$ in $\mathrm{Super}^{[n]}[\mathscr{L}_2 \mid \mathscr{P}_{x^-}]$. Write

$$\mathbf{m} = (\alpha_1 \mid x)(\alpha_2 \mid x) \cdots (\alpha_n \mid x)\left(\sum c_{ST}(S \mid \boxed{T}\,) + \sum d_{ST}(S \mid \boxed{T}\,)\right),$$

where the first sum is over standard right symmetrized bitableaux of shape λ and the second is over those of shape $\mu \neq \lambda$; and where $\mathrm{cont}(T) = \mathrm{cont}(\mathrm{Der}_-(\lambda))$ for each T. Let μ be of Frobenius type. By the assumption at the beginning,

$$0 \neq \mathscr{F}_\lambda^+ = \mathbf{U}'(\mathbf{m}) = \mathbf{U}'\left((\alpha_1 \mid x)(\alpha_2 \mid x) \cdots (\alpha_n \mid x) \sum d_{ST}(S \mid \boxed{T}\,)\right).$$

Therefore there exists some $(S \mid Q)$ of shape $\mu \neq \lambda$ such that $\mathrm{cont}(Q) = \mathrm{cont}(\mathrm{Der}_-(\lambda))$ and $(S \mid Q) \neq \mathbf{0}$ in $\mathrm{Super}^{[n]}[\mathscr{L}_2 \mid \mathscr{P}_{x^-}]$. Let $\mu = ((b'_1 - 1), (b'_2 - 1), ..., (b'_q - 1); b'_1, b'_2, ..., b'_q)$ and $\lambda = ((b_1 - 1), (b_2 - 1), ..., (b_p - 1); b_1, b_2, ..., b_p)$. Since the first row of Q has to contain $b'_1 + 1$ different symbols, we have $b'_1 + 1 \leqslant b_1 + 1$. On the other hand, the total of b_1 appearances of e_1 have to be in different rows of Q, so $b_1 \leqslant b'_1$ and hence $b_1 = b'_1$. Moreover, sorting the rows of Q, we can assume that the outmost rim of Q is

$$\begin{array}{l} e_1\ e_2 \cdots e_{b_1+1} \\ e_1 \\ \cdots \\ e_1 \end{array} .$$

Continuing similar arguments, we can prove that $p = q$ and $b_2 = b'_2, ..., b_p = b'_p$. Hence $\mu = \lambda$, which is a contradiction. ∎

To conclude, applying $\mathbf{U}'$ to (23) we get a Gordan–Capelli series for $\Lambda^n(\mathbf{S}^2(V))$:

$$\Lambda^n(\mathbf{S}^2(V)) = \bigoplus_{\substack{\lambda \vdash 2n \\ \lambda \text{ Frobenius}}} \mathbf{U}'((\alpha_1 \mid x)(\alpha_2 \mid x) \cdots (\alpha_n \mid x)\, \mathfrak{S}_{\lambda F_\lambda^+})$$

$$= \bigoplus_{\substack{\lambda \vdash 2n \\ \lambda \text{ Frobenius}}} \langle \mathbf{U}'((\alpha_1 \mid x) \cdots (\alpha_n \mid x)(F_\lambda^+ \mid \boxed{T}\,));$$

$$T \text{ is of shape } \lambda \text{ and standard on } \mathscr{P} \rangle_{\mathbb{K}}. \qquad (26)$$

This is a multiplicity-free decomposition of $\Lambda^n(\mathbf{S}^2(V))$ into $\mathrm{pl}(V)$-irreducibles, each of which is a cyclic module

$$\mathbf{U}'((\alpha_1 \mid x)(\alpha_2 \mid x)\cdots(\alpha_n \mid x)\,\mathfrak{S}_{\lambda F_\lambda^+}) = \langle \mathscr{F}_\lambda^+ \rangle_{\mathrm{pl}(V)}.$$

EXAMPLE. Let $V = V_1$. Then $\Lambda^4(\mathbf{S}^2(V)) = \Lambda^4(\Lambda^2(V))$. There are only two Frobenius type partitions of 8, which are $\lambda_1 = (5, 1, 1, 1)$ and $\lambda_2 = (4, 3, 1)$; hence

$$\Lambda^4(\Lambda^2(V)) = \langle \mathscr{F}_{\lambda_1}^+ \rangle_{\mathrm{pl}(V)} \oplus \langle \mathscr{F}_{\lambda_2}^+ \rangle_{\mathrm{pl}(V)},$$

where $\mathscr{F}_{\lambda_1}^+ = [e_1e_2][e_1e_3][e_1e_4][e_1e_5]$ and $\mathscr{F}_{\lambda_2}^+ = [e_1e_2][e_1e_3][e_1e_4][e_2e_3]$.

Next, we study the $\mathbb{Z}_2$-companion symbolic operator

$$\bar{\mathbf{U}}': \mathrm{Super}^{[n]}[\bar{\mathscr{L}}_2 \mid \bar{\mathscr{P}}_{\mathbf{x}^+}] \to \Lambda^n(\mathbf{S}^2(V)).$$

Let us start from the Gordan–Capelli series

$$\mathrm{Super}^{[n]}[\bar{\mathscr{L}}_2 \mid \bar{\mathscr{P}}_{\mathbf{x}^+}] = \bigoplus_{\lambda \vdash 2n} \quad \bigoplus_{\substack{\mathbf{S}\text{ standard on }\bar{\mathscr{P}}_2 \\ \mathrm{sh}(\mathbf{S}) = \lambda}} (\boldsymbol{\alpha}_1 \mid \mathbf{x})(\boldsymbol{\alpha}_2 \mid \mathbf{x})\cdots(\boldsymbol{\alpha}_n \mid \mathbf{x})\,\mathfrak{W}_{\lambda\mathbf{S}}.$$

If $\bar{\mathbf{U}}'((\boldsymbol{\alpha}_1 \mid \mathbf{x})(\boldsymbol{\alpha}_2 \mid \mathbf{x})\cdots(\boldsymbol{\alpha}_n \mid \mathbf{x})(\mathbf{S} \mid \boxed{\mathbf{T}}\,)) \neq 0$ for some $(\mathbf{S} \mid \boxed{\mathbf{T}}\,)$ of shape λ, then $\bar{\mathbf{U}}'((\boldsymbol{\alpha}_1 \mid \mathbf{x})(\boldsymbol{\alpha}_2 \mid \mathbf{x})\cdots(\boldsymbol{\alpha}_n \mid \mathbf{x})\,\mathfrak{W}_{\lambda\mathbf{S}})$ will be a $\mathrm{pl}(V)$-irreducible of $\Lambda^n(\mathbf{S}^2(V))$ isomorphic to the Schur module $\mathfrak{S}_{\tilde{\lambda}\mathbf{S}}$ of shape $\tilde{\lambda}$, according to the relation between Schur and Weyl modules discussed earlier. Hence by (26) the shape $\tilde{\lambda}$ has to be of Frobenius type. Moreover, let $\lambda = (b_1, b_2, ..., b_p;\ (b_1 - 1),\ (b_2 - 1), ..., (b_p - 1))$; denote by $\mathbf{F}_\lambda^-$ the conjugate tableau of $F_{\tilde{\lambda}}^+$ with the signatures of the letters flipped. Consider the tableau $\mathbf{Der}_-(\lambda)$ on $\bar{\mathscr{P}}$: $\bar{\mathbf{U}}'((\boldsymbol{\alpha}_1 \mid \mathbf{x})(\boldsymbol{\alpha}_2 \mid \mathbf{x})\cdots(\boldsymbol{\alpha}_n \mid \mathbf{x})(\mathbf{S} \mid \mathbf{Der}_-(\lambda)))$ is a linear combination of monomials $\prod [e_i^+ e_j^+]$ with the same content as $\mathrm{Coder}_+(\tilde{\lambda})$. Such a monomial is non-zero if and only if the two-element sets $\{e_i, e_j\}$ in the product are distinct, since $[e_i^+ e_j^+][e_{i'}^+ e_{j'}^+] = -[e_{i'}^+ e_{j'}^+][e_i^+ e_j^+]$ in $\Lambda^n(\mathbf{S}^2(V))$. Clearly, when $\tilde{\lambda}$ is of Frobenius type, the only such nonzero monomial is

$$\mathscr{F}_\lambda^- \stackrel{\mathrm{def}}{=} [e_1e_1][e_1e_2]\cdots[e_1e_{\lambda_1}][e_2e_2][e_2e_3]\cdots[e_2e_{\lambda_2}]$$
$$\cdots[e_pe_p][e_pe_{p+1}]\cdots[e_pe_{\lambda_p}],$$

and $\bar{\mathbf{U}}'((\boldsymbol{\alpha}_1 \mid \mathbf{x})(\boldsymbol{\alpha}_2 \mid \mathbf{x})\cdots(\boldsymbol{\alpha}_n \mid \mathbf{x})(\mathbf{F}_\lambda^- \mid \mathbf{Der}_-(\lambda))) = c'\mathscr{F}_\lambda^-$, for some $c' \in \mathbb{K}$. By a proof similar to the one of Proposition 21, we can show that $c' \neq 0$. Hence $\bar{\mathbf{U}}'((\boldsymbol{\alpha}_1 \mid \mathbf{x})(\boldsymbol{\alpha}_2 \mid \mathbf{x})\cdots(\boldsymbol{\alpha}_n \mid \mathbf{x})\,\mathfrak{W}_{\lambda\mathbf{F}_\lambda^-})$ is the $\mathrm{pl}(V)$-irreducible in $\Lambda^n(\mathbf{S}^2(V))$ isomorphic to $\mathbf{U}'((\alpha_1 \mid x)(\alpha_2 \mid x)\cdots(\alpha_n \mid x)\,\mathfrak{S}_{\tilde{\lambda}F_{\tilde{\lambda}}^+})$. From (26), we obtain

$$\Lambda^n(\mathbf{S}^2(V)) = \bigoplus_{\substack{\lambda \vdash 2n \\ \tilde{\lambda} \text{ Frobenius}}} \bar{\mathbf{U}}'((\boldsymbol{\alpha}_1 \mid \mathbf{x})(\boldsymbol{\alpha}_2 \mid \mathbf{x}) \cdots (\boldsymbol{\alpha}_n \mid \mathbf{x})\, \mathfrak{W}_{\lambda \mathbf{F}_{\tilde{\lambda}}^-})$$

$$= \bigoplus_{\substack{\lambda \vdash 2n \\ \tilde{\lambda} \text{ Frobenius}}} \langle \bar{\mathbf{U}}'((\boldsymbol{\alpha}_1 \mid \mathbf{x}) \cdots (\boldsymbol{\alpha}_n \mid \mathbf{x})(\mathbf{F}_{\tilde{\lambda}}^- \mid \boxed{\mathbf{T}}\,)); \; \mathbf{T} \text{ is of shape } \lambda \text{ and standard on } \bar{\mathscr{P}} \rangle_{\mathbb{K}}$$

$$= \bigoplus_{\substack{\lambda \vdash 2n \\ \tilde{\lambda} \text{ Frobenius}}} \langle \mathscr{F}_{\tilde{\lambda}}^- \rangle_{\mathrm{pl}(V)}. \tag{27}$$

This is the same multiplicity-free decomposition of $\Lambda^n(\mathbf{S}^2(V))$ into irreducibles as (26). The sets

$$\{\bar{\mathbf{U}}'((\boldsymbol{\alpha}_1 \mid \mathbf{x}) \cdots (\boldsymbol{\alpha}_n \mid \mathbf{x})(\mathbf{F}_{\tilde{\lambda}}^- \mid \boxed{\mathbf{T}}\,)); \mathbf{T} \text{ is of shape } \lambda \text{ and standard on } \bar{\mathscr{P}}\}$$

and

$$\{\mathbf{U}'((\alpha_1 \mid x) \cdots (\alpha_n \mid x)(F_{\tilde{\lambda}}^+ \mid \boxed{T}\,)); T \text{ is of shape } \tilde{\lambda} \text{ and standard on } \mathscr{P}\}$$

provide two different bases for the same irreducible $\langle \mathscr{F}_{\tilde{\lambda}}^- \rangle_{\mathrm{pl}(V)} = \langle \mathscr{F}_{\tilde{\lambda}}^+ \rangle_{\mathrm{pl}(V)}$.

To conclude, we have obtained

THEOREM (Gordan–Capelli Series). *The multiplicity-free decomposition of $\Lambda^n(\mathbf{S}^2(V))$ into $\mathrm{pl}(V)$-irreducibles is given by*

$$\bigoplus_{\substack{\lambda \vdash 2n \\ \lambda \text{ Frobenius}}} \langle \mathbf{U}'((\alpha_1 \mid x) \cdots (\alpha_n \mid x)(F_{\lambda}^+ \mid \boxed{T}\,)); \; T \text{ is of shape } \lambda \text{ and standard on } \mathscr{P} \rangle_{\mathbb{K}}$$

or

$$\bigoplus_{\substack{\lambda \vdash 2n \\ \tilde{\lambda} \text{ Frobenius}}} \langle \mathbf{U}'((\boldsymbol{\alpha}_1 \mid \mathbf{x}) \cdots (\boldsymbol{\alpha}_n \mid \mathbf{x})(\mathbf{F}_{\lambda}^+ \mid \boxed{\mathbf{T}}\,)); \; \mathbf{T} \text{ is of shape } \lambda \text{ and standard on } \bar{\mathscr{P}} \rangle_{\mathbb{K}}.$$

THEOREM 6 (Straightening Formulas). *Both of the sets*

$$\bigcup_{\substack{\lambda \vdash 2n \\ \lambda \text{ Frobenius}}} \{\mathbf{U}'((\alpha_1 \mid x) \cdots (\alpha_n \mid x)(F_{\lambda}^+ \mid \boxed{\mathbf{T}}\,)); \; T \text{ is of shape } \lambda \text{ and standard on } \mathscr{P}\}$$

and

$$\bigcup_{\substack{\lambda \vdash 2n \\ \tilde{\lambda} \text{ Frobenius}}} \{\mathbf{U}'((\boldsymbol{\alpha}_1 \mid \mathbf{x}) \cdots (\boldsymbol{\alpha}_n \mid \mathbf{x})(\mathbf{F}_\lambda^- \mid \boxed{\mathrm{T}}));$$

$$\mathbf{T} \textit{ is of shape } \lambda \textit{ and standard on } \mathscr{P}\}.$$

are bases of $\Lambda^n(\mathbf{S}^2(V))$.

Proof. We prove here that the first set forms a basis. It can be proved similarly that the second set also forms a basis. By earlier discussion, it is sufficient to show that

$$\langle \mathbf{U}'((\alpha_1 \mid x) \cdots (\alpha_n \mid x)(F_\lambda^+ \mid T));\ T \text{ is of shape } \lambda \text{ and standard on } \mathscr{P} \rangle_{\mathbb{K}}$$

$$= \langle \mathbf{U}'((\alpha_1 \mid x) \cdots (\alpha_n \mid x)(F_\lambda^+ \mid \boxed{T}));\ T \text{ is of shape } \lambda \text{ and standard on } \mathscr{P} \rangle_{\mathbb{K}}.$$

Straighten the element $(F_\lambda^+ \mid T)$ by right symmetrized tableaux,

$$(F_\lambda^+ \mid T) = \sum_{\substack{\mathrm{sh}(S) = \lambda \\ S \text{ standard}}} c_S(F_\lambda^+ \mid \boxed{S}) + \sum_{\substack{\mathrm{sh}(S' \mid S) > \lambda \\ (S' \mid S) \text{ standard}}} c_{S'S}(S' \mid \boxed{S}).$$

If λ is of Frobenius type, then any shape greater than λ in the dominance order cannot be of Frobenius type. Therefore applying $\mathbf{U}'$ to the preceding identity, we get by Proposition 20 that

$$\mathbf{U}'((\alpha_1 \mid x) \cdots (\alpha_n \mid x)(F_\lambda^+ \mid T))$$

$$= \mathbf{U}'\left((\alpha_1 \mid x) \cdots (\alpha_n \mid x) \sum_{\substack{\mathrm{sh}(S) = \lambda \\ S \text{ standard}}} c_S(F_\lambda^+ \mid \boxed{S})\right).$$

On the other hand, straighten $(F_\lambda^+ \mid \boxed{T})$ by standard bitableaux,

$$(F_\lambda^+ \mid \boxed{T}) = \sum_{\substack{\mathrm{sh}(S) = \lambda \\ S \text{ standard}}} d_S(F_\lambda^+ \mid S) + \sum_{\substack{\mathrm{sh}(S' \mid S) > \lambda \\ (S' \mid S) \text{ standard}}} d_{S'S}(S' \mid S).$$

Straighten the second sum by right symmetrized tableaux again,

$$(F_\lambda^+ \mid \boxed{T}) = \sum_{\substack{\mathrm{sh}(S) = \lambda \\ S \text{ standard}}} d_S(F_\lambda^+ \mid S) + \sum_{\substack{\mathrm{sh}(S' \mid S) > \lambda \\ (S' \mid S) \text{ standard}}} g_{S'S}(S' \mid \boxed{S}).$$

Hence by Proposition 20 again, we get

$$\begin{aligned}&\mathbf{U}'((\alpha_1 \mid x)\cdots(\alpha_n \mid x)(F_\lambda^+ \mid \boxed{T}))\\&=\mathbf{U}'\left((\alpha_1 \mid x)\cdots(\alpha_n \mid x) \sum_{\substack{\mathrm{sh}(S)=\lambda\\ S \text{ standard}}} d_S(F_\lambda^+ \mid S)\right). \quad \blacksquare\end{aligned}$$

7. Contragradient Actions and Umbral Calculus

In this section, we are interested in establishing umbral operators for $\mathbf{S}^n(\mathbf{S}(V)^*)$ and $\Lambda^n(\mathbf{S}(V)^*)$, where $\mathbf{S}(V)^*$ is the (*graded*) dual space of $\mathbf{S}(V)$ in the sense to be discussed next.

Let $V=V_0\oplus V_1$ be a $\mathbb{Z}_2$-graded vector space with $\dim(V)=d$; its *dual space* is defined as $V^*=\mathrm{Hom}(V,\mathbb{K})$, the set of linear functionals on V. The dual space can also be made $\mathbb{Z}_2$-graded by setting

$$V_0^*=\{f\in\mathrm{Hom}(V,\mathbb{K}),\ f\,|_{V_1}=0\}\cong(V_0)^*,$$

$$V_1^*=\{f\in\mathrm{Hom}(V,\mathbb{K}),\ f\,|_{V_0}=0\}\cong(V_1)^*.$$

Clearly $V^*=V_0^*\oplus V_1^*$. To define the notion of contragradient or adjoint action, let $\mathfrak{L}$ be a Lie superalgebra. Suppose V is an even $\mathfrak{L}$-module; i.e., $|E\cdot v|=|E|+|v|$ for all $\mathbb{Z}_2$-homogeneous elements $E\in\mathfrak{L}$ and $v\in V$. The *contragradient*, or the *adjoint*, *$\mathfrak{L}$-module* V^* is defined such that

$$(E\odot f)(v)\overset{\text{def}}{=}-(-1)^{|E|\,|f|}f(E\cdot v), \tag{28}$$

for every $E\in\mathfrak{L}$, $f\in V^*$, $v\in V$, and where $\odot$ denotes the contragradient action on V^* [49].

PROPOSITION 22. *Formula* (28) *defines a Lie superalgebra action.*

Proof. We have to check that

$$([E_1,E_2]\odot f)(v)=E_1\odot(E_2\odot f)(v)-(-1)^{|E_1|\,|E_2|}E_2\odot(E_1\odot f)(v).$$

The left side equals

$$\begin{aligned}&((E_1E_2)\odot f)(v)-(-1)^{|E_1|\,|E_2|}((E_2E_1)\odot f)(v)\\&\quad=-(-1)^{(|E_1|+|E_2|)\,|f|}f((E_1E_2)\cdot v)\\&\qquad+(-1)^{|E_1|\,|E_2|+(|E_1|+|E_2|)\,|f|}f((E_2E_1)\cdot v)\\&\quad=-(-1)^{(|E_1|+|E_2|)\,|f|}f((E_1E_2)\cdot v-(-1)^{|E_1|\,|E_2|}(E_2E_1)\cdot v)\\&\quad=-(-1)^{(|E_1|+|E_2|)\,|f|}f(E_1\cdot(E_2\cdot v)-(-1)^{|E_1|\,|E_2|}E_2\cdot(E_1\cdot v))\\&\quad=(-1)^{|E_2|\,|f|}E_1\odot f(E_2\cdot v)-(-1)^{|E_1|\,|f|+(|E_1|\,|E_2|)}E_2\odot f(E_1\cdot v)\\&\quad=-(-1)^{|E_1|\,|E_2|}E_2\odot(E_1\odot f)(v)+E_1\odot(E_2\odot f)(v). \quad\blacksquare\end{aligned}$$

Now define $\mathbf{S}(V)^* = \bigoplus_k \mathbf{S}^k(V)^*$, where $\mathbf{S}^k(V)^* \stackrel{\text{def}}{=} (\mathbf{S}^k(V))^*$, the *dual of the $\mathbb{Z}_2$-graded vector space* $\mathbf{S}^k(V)$. This defines $\mathbf{S}(V)^*$ as a proper subspace of $(\mathbf{S}(V))^*$, the dual of the entire space $\mathbf{S}(V)$. The explicit algebraic structure of $\mathbf{S}(V)^*$ will be discussed in detail later. As proved in the first section for $\mathfrak{L} = \mathrm{pl}(V)$, any $\mathfrak{L}$-module V can be extended to an $\mathfrak{L}$-module $\mathbf{S}(V)$ (or $\Lambda(V)$) by right superderivations. Hence, combining the contragradient action, any $\mathfrak{L}$-module V can be extended to an $\mathfrak{L}$-module $\mathbf{S}^n(\mathbf{S}(V)^*)$ (or $\Lambda^n(\mathbf{S}(V)^*)$). In particular, we want to study the $\mathrm{pl}(V)$-modules $\mathbf{S}^n(\mathbf{S}(V)^*)$ and $\Lambda^n(\mathbf{S}(V)^*)$. When $V = V_0$, the module $\mathbf{S}^*(\mathbf{S}^k(V)^*)$ becomes the $\mathrm{gl}(V)$-module $\mathbb{K}[\mathrm{Sym}^k(V)^*]$ of polynomials in the coordinates of a "generic" d-ary quantic of degree k, which is the subject of classical invariant theory for polynomials [19, 22–24, 45, 56, 57]. When $V = V_1$, either $\mathbf{S}(\mathbf{S}^k(V)^*)$ or $\Lambda(\mathbf{S}^k(V)^*)$ becomes $\mathbb{K}[A^k(V)^*]$ depending on k being even or odd, which is the $\mathrm{gl}(V)$-module investigated in [23] by Grosshans, Rota, and Stein. They developed a so-called *umbral method* to study the $\mathrm{sl}(V)$-invariants in $\mathbb{K}[A^k(V)]$. Our goal in this section is to establish an *umbral theory* for the $\mathrm{pl}(V)$-modules $\mathbf{S}(\mathbf{S}^k(V)^*)$ and $\Lambda(\mathbf{S}^k(V)^*)$.

We start with a general setting. Let V, V', W be $\mathbb{Z}_2$-graded vector spaces. A bilinear pairing $F\colon V \times V' \to W$ is said to be $\mathbb{Z}_2$-homogeneous whenever $|F(v, v')| = |v| + |v'|$. We denote $F(v, v')$ by $\langle v \mid v' \rangle$. The biproduct $(v \mid v')$ defined in [23] is $\mathbb{Z}_2$-homogeneous.

PROPOSITION 23. *Let V be an even $\mathfrak{L}$-module; given a non-degenerate bilinear $\mathbb{Z}_2$-homogeneous pairing $\langle \mid \rangle\colon V \times V' \to W$, set*

$$\langle v \mid \varphi \odot v' \rangle = -(-1)^{|\varphi|\,|v'|} \langle \varphi \cdot v \mid v' \rangle \tag{29}$$

for all $\mathbb{Z}_2$-homogeneous elements $\varphi \in \mathfrak{L}$, $v \in V$, $v' \in V'$. The action $\odot$ in (29) defines an even $\mathfrak{L}$-module V'.

Proof. Recall that V is an even $\mathfrak{L}$-module if and only if $|\varphi \cdot v| = |\varphi| + |v|$. Let us check $|\varphi \odot v'| = |\varphi| + |v'|$. Since $\langle \mid \rangle$ is a $\mathbb{Z}_2$-homogenous pairing, by (29) we get

$$|v| + |\varphi \odot v'| = |\varphi \cdot v| + |v'| = |\varphi| + |v| + |v'|,$$

so $|\varphi \odot v'| = |\varphi| + |v'|$. Next, check the relation

$$[\varphi, \psi] \odot v' = \varphi \odot (\psi \odot v') - (-1)^{|\varphi|\,|\psi|} \psi \odot (\varphi \odot v'). \tag{30}$$

We have

$$\begin{aligned}
&\langle v \mid [\varphi, \psi] \odot v' \rangle \\
&\quad = -(-1)^{(|\varphi| + |\psi|)\,|v'|} \langle [\varphi, \psi] \cdot v \mid v' \rangle \\
&\quad = -(-1)^{(|\varphi| + |\psi|)\,|v'|} \langle \varphi \cdot (\psi \cdot v) - (-1)^{|\varphi|\,|\psi|} \psi \cdot (\varphi \cdot v) \mid v' \rangle
\end{aligned}$$

$$= (-1)^{|\psi|\,|v'|} \langle \psi \cdot v \mid \varphi \odot v' \rangle - (-1)^{|\varphi|\,|\psi| + |\varphi|\,|v'|} \langle \varphi \cdot v \mid \psi \odot v' \rangle$$
$$= -(-1)^{|\varphi|\,|\psi|} \langle v \mid \psi \odot (\varphi \odot v') \rangle + \langle v \mid \varphi \odot (\psi \odot v') \rangle,$$

hence (30) follows, since $\langle \mid \rangle$ is bilinear and non-degenerate. ∎

The action defined in (29) will be called the *adjoint action*, or the *contragradient action*, associated to the action of $\mathfrak{L}$ on V with respect to the pairing $\langle \mid \rangle$.

Now let $\mathscr{P} = \{e_1, e_2, ..., e_d\}$ and $\mathscr{P}^* = \{f_1, f_2, ..., f_d\}$ be arbitrary $\mathbb{Z}_2$-homogeneous bases of V and V^*, respectively. Consider the biproduct

$$(\mid): \mathrm{Super}[\mathscr{P}] \times \mathrm{Super}[\mathscr{P}^*] \to \mathrm{Super}[\mathscr{P} \mid \mathscr{P}^*]$$

of Grosshans, Rota, and Stein in [23]. We know that Super[$\mathscr{P}$] is a pl(V)-module by right superderivation. Identify each element E_{ij} with the right polarization ${}_{e_j e_i}\mathcal{a}$. Denote by $\mathcal{a}$ a generic right superpolarization; any $\mathcal{a}$ on Super[$\mathscr{P}$] extends to a unique right superderivation, still denoted by $\mathcal{a}$, on Super[$\mathscr{P} \mid \mathscr{P}^*$] by setting

$$(e_i \mid f_j)\,\mathcal{a} = (-1)^{|f_j|\,|\mathcal{a}|} (e_i \mathcal{a} \mid f_j).$$

PROPOSITION 24. *We have* $(u \mid w)\,\mathcal{a} = (-1)^{|w|\,|\mathcal{a}|} (u\mathcal{a} \mid w)$, *for every* $u \in \mathrm{Super}[\mathscr{P}]$ *and* $w \in \mathrm{Super}[\mathscr{P}^*]$.

Proof. Writing $u = u_1 u_2$ and taking induction on length(u), we have

$$\begin{aligned}
&(u_1 u_2 \mid w)\,\mathcal{a} \\
&\quad = \sum_w (-1)^{|u_2|\,|w_{(1)}|} ((u_1 \mid w_{(1)})(u_2 \mid w_{(2)}))\,\mathcal{a} \\
&\quad = \sum_w (-1)^{|u_2|\,|w_{(1)}| + (|u_2| + |w_{(2)}|)\,|\mathcal{a}|} (u_1 \mid w_{(1)})\,\mathcal{a} \cdot (u_2 \mid w_{(2)}) \\
&\qquad + \sum_w (-1)^{|u_2|\,|w_{(1)}|} (u_1 \mid w_{(1)})((u_2 \mid w_{(2)})\,\mathcal{a}) \\
&\quad = \sum_w (-1)^{|u_2|\,|w_{(1)}| + |u_2|\,|\mathcal{a}| + |w|\,|\mathcal{a}|} (u_1\mathcal{a} \mid w_{(1)})(u_2 \mid w_{(2)}) \\
&\qquad + \sum_w (-1)^{|u_2|\,|w_{(1)}| + |w_{(2)}|\,|\mathcal{a}|} (u_1 \mid w_{(1)})(u_2\mathcal{a} \mid w_{(2)}) \\
&\quad = (-1)^{|w|\,|\mathcal{a}|} \Big(\sum_w (-1)^{|u_2|\,|w_{(1)}| + |u_2|\,|\mathcal{a}|} (u_1\mathcal{a} \mid w_{(1)})(u_2 \mid w_{(2)}) \\
&\qquad + \sum_w (-1)^{|w_{(1)}|\,(|u_2| + |\mathcal{a}|)} (u_1 \mid w_{(1)})(u_2\mathcal{a} \mid w_{(2)}) \Big) \\
&\quad = (-1)^{|w|\,|\mathcal{a}|} ((-1)^{|u_2|\,|\mathcal{a}|} (u_1\mathcal{a} \cdot u_2 \mid w) + (u_1 \cdot u_2\mathcal{a} \mid w)) \\
&\quad = (-1)^{|w|\,|\mathcal{a}|} ((u_1 u_2)\,\mathcal{a} \mid w). \quad ∎
\end{aligned}$$

THEOREM 7. *The adjoint action of* $\mathrm{pl}(V)$ *on* $\mathrm{Super}[\mathscr{P}^*]$, *associated with the* $\mathrm{pl}(V)$*-module* $\mathrm{Super}[\mathscr{P}]$ *with respect to the biproduct* $(\,|\,)$ *is implemented by right superderivations.*

Proof. Let $\varphi = E = \mathfrak{a}$ be a generic (right) polarization on $\mathrm{Super}[\mathscr{P}]$. Starting with the identity

$$(u \mid E \odot w) = -(-1)^{|E|\,|w|}\,(E \cdot u \mid w) = -(u \mid w)\,\mathfrak{a},$$

we have

$$\begin{aligned}
&(u \mid E \odot (w_1 w_2))\\
&\quad = -(u \mid w_1 w_2)\,\mathfrak{a}\\
&\quad = -\sum_u (-1)^{|w_1|\,|u_{(2)}|}\,((u_{(1)} \mid w_1)(u_{(2)} \mid w_2))\,\mathfrak{a}\\
&\quad = -\sum_u (-1)^{|w_1|\,|u_{(2)}| + |u_{(2)}|\,|\mathfrak{a}| + |w_2|\,|\mathfrak{a}|}\,((u_{(1)} \mid w_1)\,\mathfrak{a})(u_{(2)} \mid w_2)\\
&\qquad -\sum_u (-1)^{|w_1|\,|u_{(2)}|}\,(u_{(1)} \mid w_1)((u_{(2)} \mid w_2)\,\mathfrak{a})\\
&\quad = \sum_u (-1)^{|w_1|\,|u_{(2)}| + |u_{(2)}|\,|E| + |w_2|\,|E|}\,(u_{(1)} \mid E \odot w_1)(u_{(2)} \mid w_2)\\
&\qquad +\sum_u (-1)^{|w_1|\,|u_{(2)}|}\,(u_{(1)} \mid w_1)(u_{(2)} \mid E \odot w_2)\\
&\quad = (-1)^{|w_2|\,|E|}\,(u \mid (E \odot w_1)\,w_2) + (u \mid w_1(E \odot w_2)),
\end{aligned}$$

hence

$$E \odot (w_1 w_2) = (-1)^{|w_2|\,|E|}\,(E \odot w_1)\,w_2 + w_1(E \odot w_2),$$

since $(\,|\,)$ is bilinear and non-degenerate. ∎

Now let $F\colon \mathrm{Super}[\mathscr{P} \mid \mathscr{P}^*] \to \mathbb{K}$ be the linear map extended by

$$F((e_{i_1} \mid f_{j_1})(e_{i_2} \mid f_{j_2}) \cdots (e_{i_k} \mid f_{j_k})) = f_{j_1}(e_{i_1}) f_{j_2}(e_{i_2}) \cdots f_{j_k}(e_{i_k}).$$

Since $f_j(e_i) \in \mathbb{K} = \mathbb{K}_0$, we have $0 = |f_j(e_i)| = |f_j| + |e_i|$; so $f_j(e_i) = 0$ whenever $|f_j| \neq |e_i|$ and the map F preserves the $\mathbb{Z}_2$-grading. Moreover, the pairing

$$\langle\,|\,\rangle\colon \mathrm{Super}[\mathscr{P}] \times \mathrm{Super}[\mathscr{P}^*] \to \mathbb{K}$$

given by

$$\langle u \mid w \rangle = F((u \mid w)) \tag{31}$$

is $\mathbb{Z}_2$-homogeneous. Clearly, $\langle \mid \rangle$ is a non-degenerate bilinear map, even when restricted on $\mathrm{Super}^k[\mathscr{P}] \times \mathrm{Super}^k[\mathscr{P}^*]$, where $\mathrm{Super}^k[\mathscr{P}]$ and $\mathrm{Super}^k[\mathscr{P}^*]$ are the subspaces spanned by monomials of length k. Hence (31) establishes an isomorphism between the vector spaces $(\mathrm{Super}^k[\mathscr{P}])^*$ and $\mathrm{Super}^k[\mathscr{P}^*]$, and thus between the vector spaces $\mathbf{S}^k(V)^*$ and $\mathbf{S}^k(V^*)$.

From now on, we specialize the sets $\mathscr{P}$ and $\mathscr{P}^*$ to be *dual bases* of V and V^*; that is, $f_j(e_i) = \delta_{ji}$. In particular we have $|f_i| = |e_i|$, which is sometimes denoted as $|i|$. To state the next result, we recall the notion of *super-transposition* of Scheunert [49]. Define a map C: $\mathrm{pl}(V) \to \mathrm{pl}(V)$ by setting

$$C(E_{ij}) = -(-1)^{|i| + |j|\,|i|} E_{ji}.$$

PROPOSITION 25. *The map C is an even Lie superalgebra isomorphism.*

Proof. It is clear that C is even, since $|E_{ij}| = |E_{ji}| = |i| + |j|$. To check the relation

$$C([E_{ij}, E_{hk}]) = [C(E_{ij}), C(E_{hk})],$$

we have that the left side equals

$$\begin{aligned} &C(\delta_{jh} E_{ik} - (-1)^{(|i|+|j|)(|h|+|k|)} \delta_{ik} E_{hj}) \\ &\quad = -(-1)^{|i| + |i|\,|k|} \delta_{jh} E_{ki} + (-1)^{|i|\,|h| + |i|\,|k| + |j|\,|k| + |h|} \delta_{ik} E_{jh}. \end{aligned}$$

On the other hand, the right side equals

$$\begin{aligned} &[-(-1)^{|i| + |i|\,|j|} E_{ji}, -(-1)^{|h| + |h|\,|k|} E_{kh}] \\ &\quad = (-1)^{|i| + |h| + |i|\,|j| + |h|\,|k|} \delta_{ik} E_{jh} \\ &\qquad - (-1)^{|i| + |h| + |i|\,|j| + |h|\,|k| + |i|\,|k| + |i|\,|h| + |j|\,|k| + |j|\,|h|} \delta_{jh} E_{ki}. \end{aligned}$$

Because $\delta_{ij} = 0$ unless $i \neq j$, the two sides are the same. ∎

THEOREM 8. *The adjoint action of* $\mathrm{pl}(V)$ *on* $\mathrm{Super}[\mathscr{P}^*]$ *associated with the* $\mathrm{pl}(V)$*-module* $\mathrm{Super}[\mathscr{P}]$ *with respect to the biproduct* $\langle \mid \rangle$ *is implemented by the right superderivations such that*

$$\begin{aligned} E_{ij} \odot f_k = -(-1)^{|i| + |i|\,|j|} E_{ji}^* \cdot f_k &= -(-1)^{|i| + |i|\,|j|} (f_k)_{f_i f_j} \mathfrak{a} \\ &= -(-1)^{|i| + |i|\,|j|} \delta_{ik} f_j, \end{aligned} \tag{32}$$

where E_{ji}^* is the linear transformation of V^* such that $E_{ji}^*(f_k) = \delta_{ik} f_j$; in other words the action $E_{ij} \odot f_k$ is accomplished by supertransposition.

Proof. The fact that the adjoint action of $\mathrm{pl}(V)$ on $\mathrm{Super}[\mathscr{P}^*]$ with respect to $\langle \mid \rangle$ is a right superderivation immediately follows from the previous theorem. To see (32) we have

$$\begin{aligned}\langle e_h \mid E_{ij} \odot f_k \rangle &= -(-1)^{(|i|+|j|)|k|} \langle E_{ij} \cdot e_h \mid f_k \rangle \\ &= -(-1)^{(|i|+|j|)|k|} \delta_{jh} \langle e_i \mid f_k \rangle \\ &= -(-1)^{(|i|+|j|)|k|} \delta_{jh} \delta_{ik} \\ &= -(-1)^{(|i|+|j|)|k|} \delta_{ik} \langle e_h \mid f_j \rangle.\end{aligned}$$

Hence formula (32) holds. ∎

Next we give a combinatorial presentation of $\mathbf{S}^n(\mathbf{S}^k(V)^*)$ and $\Lambda^n(\mathbf{S}^k(V)^*)$. Let us define the map $K: \mathbf{S}^k(V^*) \to \mathbf{S}^k(V)^*$ such that

$$K(f_{i_1} f_{i_2} \cdots f_{i_k}) = f_{i_1 i_2 \cdots i_k},$$

where $f_{i_1 i_2 \cdots i_k}$ is the function on $\mathbf{S}^k(V)$ such that

$$f_{i_1 i_2 \cdots i_k}(e_{j_1} e_{j_2} \cdots e_{j_k}) = \langle e_{j_1} e_{j_2} \cdots e_{j_k} \mid f_{i_1} f_{i_2} \cdots f_{i_k} \rangle. \tag{33}$$

PROPOSITION 26. *The map K is a* $\mathrm{pl}(V)$*-isomorphism.*

Proof. It is well known that K is an isomorphism between the two vector spaces. So we only have to check that K is equivariant; i.e.,

$$\begin{aligned} &K(\varphi \odot (f_{i_1} f_{i_2} \cdots f_{i_k}))(e_{j_1} e_{j_2} \cdots e_{j_k}) \\ &\quad = \varphi \odot K(f_{i_1} f_{i_2} \cdots f_{i_k})(e_{j_1} e_{j_2} \cdots e_{j_k}), \end{aligned} \tag{34}$$

for $\varphi \in \mathrm{pl}(V)$. The left side, by (29) and (33), equals

$$\begin{aligned} &\langle e_{j_1} e_{j_2} \cdots e_{j_k} \mid \varphi \odot (f_{i_1} f_{i_2} \cdots f_{i_k}) \rangle \\ &\quad = -(-1)^{|\varphi|\,|f_{i_1} f_{i_2} \cdots f_{i_k}|} \langle \varphi \cdot (e_{j_1} e_{j_2} \cdots e_{j_k}) \mid f_{i_1} f_{i_2} \cdots f_{i_k} \rangle. \end{aligned}$$

The right side of (34), by (28) and (33), equals

$$\begin{aligned} &-(-1)^{|\varphi|\,|f_{i_1} f_{i_2} \cdots f_{i_k}|} K(f_{i_1} f_{i_2} \cdots f_{i_k})(\varphi \cdot (e_{j_1} e_{j_2} \cdots e_{j_k})) \\ &\quad = -(-1)^{|\varphi|\,|f_{i_1} f_{i_2} \cdots f_{i_k}|} \langle \varphi \cdot (e_{j_1} e_{j_2} \cdots e_{j_k}) \mid f_{i_1} f_{i_2} \cdots f_{i_k} \rangle. \quad ∎ \end{aligned}$$

Clearly, by definition

(i) $f_{i_1 i_2 \cdots i_p i_{p+1} \cdots i_k} = (-1)^{|i_p|\,|i_{p+1}|} f_{i_1 i_2 \cdots i_{p+1} i_p \cdots i_k}$, $p = 1, 2, ..., k-1$;

(ii) if $V_0 = \langle e_1, ..., e_{d_0} \rangle_{\mathbb{K}}$ and $V_1 = \langle e_{d_0+1}, ..., e_d \rangle_{\mathbb{K}}$, then

$$\begin{aligned} f_{1^{m_1} 2^{m_2} \cdots d^{m_d}}(e_1^{n_1} e_2^{n_2} \cdots e_d^{n_d}) &= \langle e_1^{n_1} e_2^{n_2} \cdots e_d^{n_d} \mid f_1^{m_1} f_2^{m_2} \cdots f_d^{m_d} \rangle \\ &= (-1)^{\binom{k-m}{2}} m_1!\, m_2! \cdots m_{d_0}!\, \delta_{n_1 m_1} \delta_{n_2 m_2} \cdots \delta_{n_d m_d}, \end{aligned}$$

where $m = m_1 + m_2 + \cdots + m_{d_0} \leqslant k$, and $m_{d_0+i} = 1$ or 0 for $i = 1, 2, ..., d - d_0$.

The algebraic structure of $\mathbf{S}(\mathbf{S}^k(V)^*)$ can be made explicit as follows. Let $[V]_k^*$ be the set $\{f_{i_1 i_2 \cdots i_k};\ 1 \leqslant i_1, i_2, ..., i_k \leqslant d\}$. Denote by $\mathrm{Mon}([V]_k^*)$ the free monoid generated by $[V]_k^*$ and form the semigroup algebra $\mathbb{K}[\mathrm{Mon}([V]_k^*)]$. Let I be the ideal of $\mathbb{K}[\mathrm{Mon}([V]_k^*)]$ generated by elements of the types

(i) $f_{i_1 i_2 \cdots i_p i_{p+1} \cdots i_k} - (-1)^{|i_p|\,|i_{p+1}|} f_{i_1 i_2 \cdots i_{p+1} i_p \cdots i_k}$;

(ii) $f_{i_1 i_2 \cdots i_k} f_{j_1 j_2 \cdots j_k} - (-1)^{(|i_1| + \cdots + |i_k|)(|j_1| + \cdots + |j_k|)} f_{j_1 j_2 \cdots j_k} f_{i_1 i_2 \cdots i_k}$.

PROPOSITION 27. $\mathbf{S}(\mathbf{S}^k(V)^*) \cong \mathbb{K}[\mathrm{Mon}([V]_k^*)]/I$.

The explicit algebraic structure of $\Lambda(\mathbf{S}^k(V)^*)$ can be given similarly, except that one should have the expression

(ii′) $f_{i_1 i_2 \cdots i_k} f_{j_1 j_2 \cdots j_k} - (-1)^{(|i_1| + \cdots + |i_k| + 1)(|j_1| + \cdots + |j_k| + 1)} f_{j_1 j_2 \cdots j_k} f_{i_1 i_2 \cdots i_k}$

to replace (ii).

The identification K between $\mathbf{S}^k(V^*)$ and $\mathbf{S}^k(V)^*$ explains the phenomenon that *indices become exponents* in the umbral method of classical invariant theory for polynomials; to make this apparent, keeping the earlier notation, note that a generic element in $\mathbf{S}^k(V)$ is a $\mathbb{K}$-homogeneous "polynomial" in the $\mathbb{Z}_2$-graded supercommuting variables $e_1^+, e_2^+, ..., e_{d_0}^+$, $e_{d_0+1}^-, ..., e_d^-$, which can be written as

$$t = \sum_m \sum \binom{m}{m_1 m_2 \cdots m_{d_0}} t_{m_1 m_2 \cdots m_d}(t)\, e_1^{m_1} e_2^{m_2} \cdots e_d^{m_d},$$

where the outer sum ranges over all m such that $0 \leqslant m \leqslant k$, and the inner sum is over the d-tuples $(m_1, m_2, ..., m_d)$ such that $m_1 + m_2 + \cdots + m_d = k$, $m_1 + m_2 + \cdots + m_{d_0} = m$, and $0 \leqslant m_{d_0+i} \leqslant 1$, $i = 1, 2, ..., d - d_0$. We have

$$\begin{aligned} f_{1^{m_1} 2^{m_2} \cdots d^{m_d}}(t) &= \langle t \mid f_1^{m_1} f_2^{m_2} \cdots f_d^{m_d} \rangle \\ &= \binom{m}{m_1 m_2 \cdots m_{d_0}} t_{m_1 m_2 \cdots m_d}(t)\, \langle e_1^{m_1} e_2^{m_2} \cdots e_d^{m_d} \mid f_1^{m_1} f_2^{m_2} \cdots f_d^{m_d} \rangle \\ &= (-1)^{\binom{k-m}{2}}\, m!\, t_{m_1 m_2 \cdots m_d}(t). \end{aligned}$$

When $V = V_0$, everything is positive; we can treat $m!\, t_{m_1 m_2 \cdots m_d}$ as $f_1^{m_1} f_2^{m_2} \cdots f_d^{m_d} = K^{-1}(f_{1^{m_1} 2^{m_2} \cdots d^{m_d}})$, which is exactly the main idea of the umbral method (up to a global coefficient $m!$) used in classical invariant theory of d-ary forms of degree k (see [19, 22, 23, 36, 44, 53, 56, 57]).

Now we define the *umbral linear operators*

$$\mathscr{U}\colon \mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}^*_{x+}] \to \mathbf{S}^n(\mathbf{S}^k(V)^*),$$

$$\mathscr{U}'\colon \mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}^*_{x-}] \to \Lambda^n(\mathbf{S}^k(V)^*),$$

$$\bar{\mathscr{U}}\colon \mathrm{Super}^{[n]}[\bar{\mathscr{L}}_k \mid \bar{\mathscr{P}}^*_{\mathbf{x}-}] \to \mathbf{S}^n(\mathbf{S}^k(V)^*),$$

$$\bar{\mathscr{U}}'\colon \mathrm{Super}^{[n]}[\bar{\mathscr{L}}_k \mid \bar{\mathscr{P}}^*_{\mathbf{x}+}] \to \Lambda^n(\mathbf{S}^k(V)^*),$$

by $\mathscr{U} = K \circ \mathbf{U}$, $\mathscr{U}' = K \circ \mathbf{U}'$, $\bar{\mathscr{U}} = K \circ \bar{\mathbf{U}}$, $\bar{\mathscr{U}}' = K \circ \bar{\mathbf{U}}'$, where $\mathbf{U}$, $\mathbf{U}'$, $\bar{\mathbf{U}}$, and $\bar{\mathbf{U}}'$ are the symbolic operators for $\mathbf{S}^n(\mathbf{S}^k(V^*))$ and $\Lambda^n(\mathbf{S}^k(V^*))$ defined in (5), (6), (8), and (9), except that each e_i is replaced by f_i. We remark that the umbral operator $\mathscr{U}$ for $\mathbf{S}^n(\mathbf{S}^k(V)^*) = \mathrm{Sym}^n(A^k(V)^*)$ when $V = V_1$, k even, together with the umbral operator $\mathscr{U}'$ for $\Lambda^n(\mathbf{S}^k(V)^*) = \mathrm{Sym}^n(A^k(V)^*)$ when $V = V_1$, k odd, become the umbral operator of Grosshans, Rota, and Stein for $\mathrm{Sym}^n(A^k(V)^*)$, $k \in \mathbb{N}$, studied in [23].

Note that $\mathbf{S}^n(\mathbf{S}^k(V)^*)$ and $\Lambda^n(\mathbf{S}^k(V)^*)$ are $\mathrm{pl}(V)$-modules; so are the letterplace algebras $\mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}^*_x]$ and $\mathrm{Super}^{[n]}[\bar{\mathscr{L}}_k \mid \bar{\mathscr{L}}^*_{\mathbf{x}}]$, where the actions are the right superderivations such that

$$E_{ij} \cdot (\alpha \mid f) = (\alpha \mid E_{ij} \odot f) = -(-1)^{|i| + |i|\,|j|}\,(\alpha \mid f\;{}_{f_i f_j}a),$$

for $\alpha \in \mathscr{L}$, $f \in \mathscr{P}^*_x$; or $\boldsymbol{\alpha} \in \bar{\mathscr{L}}$, $\mathbf{f} \in \bar{\mathscr{P}}^*_{\mathbf{x}}$.

THEOREM 9. *The umbral operators $\mathscr{U}$, $\mathscr{U}'$, $\bar{\mathscr{U}}$, and $\bar{\mathscr{U}}'$ are well defined* $\mathrm{pl}(V)$*-epimorphisms.*

Proof. The assertion follows immediately from the earlier results about $\mathbf{U}$, $\mathbf{U}'$, $\bar{\mathbf{U}}$, $\bar{\mathbf{U}}'$, and K. ▮

Since the $\mathrm{pl}(V)$-action on $\mathbf{S}(\mathbf{S}^k(V^*))$ is given by the supertransposition of the $\mathrm{pl}(V^*)$-action on $\mathbf{S}(\mathbf{S}^k(V^*))$ in the sense of (32), the complete decomposition of $\mathbf{S}(\mathbf{S}^k(V^*))$ into irreducibles is the same whether it is considered as a $\mathrm{pl}(V)$-module or as a $\mathrm{pl}(V^*)$-module. Such a decomposition simultaneously provides a complete decomposition of $\mathbf{S}(\mathbf{S}^k(V)^*)$ into $\mathrm{pl}(V)$-irreducibles via the isomorphism K. Similar statements can be made for $\Lambda(\mathbf{S}^k(V)^*)$. Therefore the Gordan–Capelli series and the straightening formulas of $\mathbf{S}(\mathbf{S}^k(V)^*)$ and $\Lambda(\mathbf{S}^k(V)^*)$ can also be obtained from the ones for $\mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}^*_x]$ and $\mathrm{Super}^{[n]}[\bar{\mathscr{L}}_k \mid \bar{\mathscr{P}}^*_{\mathbf{x}}]$ under the suitable umbral operators.

EXAMPLE (Binary Forms). Let $V = V_0$, $d = 2$; then $m = k$. A generic element in $\mathbf{S}^k(V) = \mathrm{Sym}^k(V)$ is the binary form

$$t = \sum_{i+j=k} \binom{k}{i} t_{ij}(t)\, e_1^i e_2^j.$$

In the classical umbral method, t_{ij} is considered as $(1/k!)f_1^i f_2^j$, where the indices i, j are raised to exponents. To be precise, in our setting, $\mathscr{U} \cdot (\alpha^{(k)} \mid f_1^i f_2^j) = k!\, t_{ij}$, where t_{ij} is regarded as the linear functional on binary forms such that $t_{ij}(e_1^h e_2^{k-h}) = \binom{k}{i}^{-1} \delta_{ih} \delta_{j\, k-h}$. Let us do an interesting check here for the action of E_{12}. We have

$$E_{12} \cdot t = \sum_{i+j=k} j \binom{k}{i} t_{ij} e_1^{i+1} e_2^{j-1} = \sum_{i+j=k} i \binom{k}{i} t_{i-1\, j+1} e_1^i e_2^j.$$

Therefore

$$E_{12} \odot t_{ij} = -i t_{i-1\, j+1}. \tag{35}$$

On the other hand

$$E_{12} \odot (f_1^i f_2^j) = -(f_1^i f_2^j)_{f_1 f_2} \mathfrak{a} = -i f_1^{i-1} f_2^{j+1}, \tag{36}$$

by Theorem 8. Comparing (35) and (36), the "identification" between t_{ij} and $(1/k!)f_1^i f_2^j$ is indeed *good.* ∎

8. The First Fundamental Theorem

In terms of representation theory, the first fundamental theorem of classical invariant theory predicts certain forms for the one-dimensional irreducible submodules. In this section, we derive a concrete description of any irreducible submodule of $\mathbf{S}^n(\mathbf{S}^k(V))$, $\Lambda^n(\mathbf{S}^k(V))$, $\mathbf{S}^n(\mathbf{S}^k(V)^*)$, and $\Lambda^n(\mathbf{S}^k(V)^*)$. First, by a tableau on $\mathscr{L}_k$ we shall mean a tableau on $\mathscr{L}$ such that each of symbolic letters $\alpha_1, \alpha_2, ..., \alpha_n$ appears exactly k times in it.

We start with the Gordan–Capelli series

$$\mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}] = \bigoplus_{\lambda \vdash kn} \bigoplus_{\substack{S \text{ standard on } \mathscr{L}_k \\ \mathrm{sh}(S) = \lambda}} \mathfrak{S}_{\lambda S},$$

where $\mathfrak{S}_{\lambda S}$ is the Schur module

$$\mathfrak{S}_{\lambda S} = \langle (S \mid \boxed{T}\,);\ T \text{ is of shape } \lambda \text{ and standard on } \mathscr{P} \rangle_{\mathbb{K}}.$$

Similarly

$$\mathrm{Super}^{[n]}[\bar{\mathscr{L}}_k \mid \bar{\mathscr{P}}] = \bigoplus_{\lambda \vdash kn} \bigoplus_{\substack{\mathbf{S} \text{ standard on } \bar{\mathscr{L}}_k \\ \mathrm{sh}(\mathbf{S}) = \lambda}} \mathfrak{W}_{\lambda \mathbf{S}},$$

where $\mathfrak{W}_{\lambda \mathbf{S}}$ is the Weyl module

$$\mathfrak{W}_{\lambda \mathbf{S}} = \langle (\mathbf{S} \mid \boxed{\mathbf{T}}\,);\ \mathbf{T} \text{ is of shape } \lambda \text{ and standard on } \bar{\mathscr{P}} \rangle_{\mathbb{K}}.$$

PROPOSITION 28. *Let M be a* $\mathrm{pl}(V)$*-irreducible submodule of* $\mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}]$. *Then there exists an element* $\mathbf{m}$ *of the form*

$$\mathbf{m} = \sum_{\substack{S \text{ standard on } \mathscr{L}_k \\ \mathrm{sh}(S) = \lambda}} c_s(S \mid \mathrm{Der}_-(\lambda)) \tag{37}$$

for some shape $\lambda \vdash kn$, such that $M = \langle \mathbf{m} \rangle_{\mathrm{pl}(V)}$. Similar statement holds for $\mathrm{Super}^{[n]}[\bar{\mathscr{L}}_k \mid \bar{\mathscr{P}}]$.

Remark. The formula (37) holds in general as long as $d_1 = \dim(V_1)$ is large enough to form $\mathrm{Der}_-(\lambda)$. If d_1 is not large enough for a particular V, we can replace $\mathrm{Der}_-(\lambda)$ in (37) by a right symmetrized standard tableau $\boxed{T}$ to make the proposition true.

Proof. Without loss of generality, we shall assume that both d_1 and d_0 are large enough. Let $\mathbf{m}$ be a non-zero element in M and let

$$\mathbf{m} = \sum_{\lambda, S} m_{\lambda S}, \tag{38}$$

where $m_{\lambda S}$ is the projection of $\mathbf{m}$ into the irreducible $\mathfrak{S}_{\lambda S}$. Note that the projection $\mathbf{m} \to m_{\lambda S}$ is a $\mathrm{pl}(V)$-equivariant map, so $\langle m_{\lambda S} \rangle_{\mathrm{pl}(V)}$ is either zero or the irreducible $\mathfrak{S}_{\lambda S}$ by the first Schur Lemma, depending on $m_{\lambda S}$ being zero or not. Clearly, $M \cong \langle m_{\lambda S} \rangle_{\mathrm{pl}(V)} = \mathfrak{S}_{\lambda S}$ whenever $m_{\lambda S} \neq 0$. Therefore only one shape can appear in (38) and we can write

$$\mathbf{m} = \sum_S m_{\lambda S} \tag{39}$$

for some λ. Assume $m_{\lambda S} \neq 0$ for all terms in (39) and fix a particular S. Since $m_{\lambda S}$ generate the irreducible $\mathfrak{S}_{\lambda S}$, we have $\varphi \cdot m_{\lambda S} = (S \mid \boxed{\mathrm{Der}_-(\lambda)}\,) = c \cdot (S \mid \mathrm{Der}_-(\lambda))$, $c \neq 0$, for some $\varphi \in \mathrm{pl}(V)$. Hence without loss of generality, we may assume that $m_{\lambda S} = (S \mid \mathrm{Der}_-(\lambda))$. We claim $m_{\lambda S'} = c_{S'}(S' \mid \mathrm{Der}_-(\lambda))$ for all S' in (39). To see it, first note that the map

$$\varphi \cdot m_{\lambda S} \xrightarrow{F} \varphi \cdot m_{\lambda S'}$$

is a $\mathrm{pl}(V)$-isomorphism between $\langle m_{\lambda S} \rangle_{\mathrm{pl}(V)}$ and $\langle m_{\lambda S'} \rangle_{\mathrm{pl}(V)}$.

Let $\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, ..., \tilde{\lambda}_p)$ and let

$$\varphi_i = (E_{i^- \beta^-})^{\tilde{\lambda}_i} (E_{\beta^- i^-})^{\tilde{\lambda}_i}$$

for some (virtual) place $\beta^- \in \mathscr{P} \backslash \{e_1^-, e_2^-, ..., e_{\tilde{\lambda}_1}^-\}$. The action of φ_i on $\mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}]$ has the property

$$\varphi_i \cdot (S' \mid \boxed{T}\,) = \begin{cases} c \cdot (S' \mid \boxed{T}\,), \text{ for some } c \neq 0 & \text{if } \mathrm{cont}(T, e_i) \geqslant \tilde{\lambda}_i, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\varphi_i \cdot (S \mid \mathrm{Der}_-(\lambda)) = \varphi_i \cdot m_{\lambda S}$ is a non-zero multiple of $m_{\lambda S}$ for each i; so must be the action $\varphi_i \cdot m_{\lambda S'}$. Therefore if

$$m_{\lambda S'} = \sum_{\substack{T \text{ standard on } \mathscr{P} \\ \mathrm{sh}(T)=\lambda}} c_{S'T}(S' \mid \boxed{T}), \tag{40}$$

then $\mathrm{cont}(T, e_i) \geqslant \tilde{\lambda}_i$ for all $i = 1, 2, ..., p$. Hence $\mathrm{cont}(T, e_i) = \tilde{\lambda}_i$ for every i, since $\mathrm{sh}(T) = \lambda$. Therefore T has to be $\mathrm{Der}_-(\lambda)$, since T is standard; hence, only one term appears in (40). So

$$m_{\lambda S'} = c_{S', \mathrm{Der}_-(\lambda)}(S' \mid \mathrm{Der}_-(\lambda)) = c_{S'}(S' \mid \mathrm{Der}_-(\lambda)).$$

Thus $\mathbf{m} = \sum_{S'} c_{S'}(S' \mid \mathrm{Der}_-(\lambda))$ as desired. ▌

An alternative proof may be easily derived from Theorem 2 of [9]. From Proposition 28, we infer

THEOREM 10 (The First Fundamental Theorem). *Let $\mathfrak{M}$ be an irreducible $\mathrm{pl}(V)$-module of $\mathbf{S}^n(\mathbf{S}^k(V))$. There exists an element $\mathbf{m} \in \mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}_x]$ of the form*

$$\mathbf{m} = (\alpha_1 \mid x)(\alpha_2 \mid x) \cdots (\alpha_n \mid x) \sum_{\substack{S \text{ standard on } \mathscr{L}_k \\ \mathrm{sh}(S)=\lambda}} c_S(S \mid \mathrm{Der}_-(\lambda)) \tag{41}$$

for some $\lambda \vdash kn$, such that

$$\mathfrak{M} = \langle \mathbf{U}(\mathbf{m}) \rangle_{\mathrm{pl}(V)}; \tag{42}$$

or more explicitly

$$\mathfrak{M} = \left\langle \mathbf{U}\left((\alpha_1 \mid x) \cdots (\alpha_n \mid x) \sum_S c_S(S \mid \boxed{T}) \right); \; T \text{ is of shape } \lambda \text{ and standard on } \mathscr{P} \right\rangle_{\mathbb{K}}. \tag{43}$$

Moreover, if $\mathfrak{M}$ is non-zero, then the set $\{\mathbf{U}((\alpha_1 \mid x) \cdots (\alpha_n \mid x) \sum_S c_S(S \mid \boxed{T}));$ T is of shape λ and standard on $\mathscr{P}\}$ is a basis of $\mathfrak{M}$. Similar statements can be made for the other operators $\mathbf{U}'$, $\bar{\mathbf{U}}$, $\bar{\mathbf{U}}'$ and $\mathscr{U}$, $\mathscr{U}'$, $\bar{\mathscr{U}}$, $\bar{\mathscr{U}}'$. Vice versa, if $\mathfrak{M} = \langle \mathbf{U}(\mathbf{m}) \rangle_{\mathrm{pl}(V)}$, where $\mathbf{m}$ is of the form (41), then $\mathfrak{M}$ is irreducible. ▌

Again, this theorem is true in general as long as d_0 or d_1 is large enough to form $\mathrm{Der}_-(\lambda)$ on $\mathscr{P}$ or $\mathrm{Der}_-(\lambda)$ on $\bar{\mathscr{P}}$. Otherwise, one has to replace $\mathrm{Der}_-(\lambda)$ in (41) by some $\boxed{T}$. In any circumstances, formula (43) always holds. The assumption we have made about the large supply of negative symbols in $\mathscr{P}$ or $\bar{\mathscr{P}}$ will be made clearer when we discuss the second fundamental theorem in next section.

To see that Theorem 10 is indeed a generalization of the first fundamental theorem of classical invariant theory, let us consider the case $V = V_0$ and the symbolic operator

$$\bar{\mathbf{U}}\colon \mathrm{Super}^{[n]}[\mathscr{L}_k \mid \bar{\mathscr{P}}_{\mathbf{x}^-}] \to \mathrm{Sym}^n(\mathrm{Sym}^k(V)).$$

The absolute $\mathbf{SL}(V)$-invariants, or the relative $\mathbf{GL}(V)$-invariants, of $\mathrm{Sym}^n(\mathrm{Sym}^k(V))$ form the isotypic component of $\mathrm{Sym}^n(\mathrm{Sym}^k(V))$ consisting of the one-dimensional irreducibles. On the other hand, any such an irreducible of $\mathrm{Super}^{[n]}[\mathscr{L}_k \mid \bar{\mathscr{P}}_{\mathbf{x}^-}]$ has to be $\mathbb{K}\mathbf{m}$, where $\mathbf{m}$ is of the form

$$\mathbf{m} = (\boldsymbol{\alpha}_1 \mid \mathbf{x})(\boldsymbol{\alpha}_2 \mid \mathbf{x}) \cdots (\boldsymbol{\alpha}_n \mid \mathbf{x}) \sum_{\mathrm{S\ standard\ on\ } \mathscr{L}_k} c_{\mathrm{S}} \left(\mathrm{S} \left| \begin{matrix} \mathbf{e}_1 & \mathbf{e}_2 \cdots \mathbf{e}_d \\ \mathbf{e}_1 & \mathbf{e}_2 \cdots \mathbf{e}_d \\ \mathbf{e}_1 & \mathbf{e}_2 \cdots \mathbf{e}_d \end{matrix} \right. \right),$$

according to Proposition 28, since $\lambda = (d, d, ..., d)$ is the only shape which allows *one and only one* standard tableau on $\bar{\mathscr{P}} = \{\mathbf{e}_1^-, \mathbf{e}_2^-, ..., \mathbf{e}_d^-\}$. Therefore, Theorem 10 implies that $\mathrm{Sym}^n(\mathrm{Sym}^k(V))$ has invariants if and only if $d \mid kn$ and any such invariant is of the form

$$\bar{\mathbf{U}}(\mathbf{m}) = \sum c_{\omega_1 \omega_2 \cdots} \bar{\mathbf{U}}((\boldsymbol{\omega}_1 \mid \mathbf{e}_1 \mathbf{e}_2 \cdots \mathbf{e}_d)(\boldsymbol{\omega}_2 \mid \mathbf{e}_1 \mathbf{e}_2 \cdots \mathbf{e}_d) \cdots);$$

this assertion is indeed the first fundamental theorem for invariants of $\mathrm{Sym}^n(\mathrm{Sym}^k(V))$.

An important fact should be noted, that when both $d_0 \neq 0$ and $d_1 \neq 0$ there is not a "strict" analog of the notion of a relative $\mathbf{GL}(V)$-invariant, since $\mathbf{S}^n(\mathbf{S}^k(V))$ has no one-dimensional $\mathrm{pl}(V)$-irreducibles. To see this, consider a Schur module

$$\mathfrak{S}_{\lambda S} = \langle (S \mid \boxed{T}\,);\ T \text{ is of shape } \lambda \text{ and standard on } \mathscr{P} \rangle_{\mathbb{K}}$$

in $\mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}]$, where $\lambda_{d_0+1} \leqslant d_1$ according to the hook condition of Berele and Regev [6]. Take a linear order

$$e_1'^{\,+} < e_2'^{\,+} < \cdots < e_{d_0}'^{\,+} < e_1^- < e_2^- < \cdots < e_{d_1}^-,$$

where $\mathscr{P}^+ = \{e_1', e_2', ..., e_{d_0}'\}$ and $\mathscr{P}^- = \{e_1, e_2, ..., e_{d_1}\}$. Let $\mu = (\lambda_1, \lambda_2, ..., \lambda_{d_0})$ and $\nu = (\lambda_{d_0+1}, \lambda_{d_0+2}, ...)$. Let T be the standard tableau obtained by putting the tableau $\mathrm{Der}_-(\nu; e_1, e_2, ..., e_{d_1})$ under the tableau $\mathrm{Coder}_+(\mu; e_1', e_2', ..., e_{d_0}')$. If $\lambda_{d_0} = 0$, then replace the last letter e_i' in the last row of T by e_1^-; if $\lambda_{d_0} > 0$, we replace the last e_{d_0}' in d_0th row of T by e_1^-. In both cases, we get another standard tableau on $\mathscr{P}$ other than T with shape λ. So $\dim(\mathfrak{S}_{\lambda S}) \geqslant 2$ for any Schur module. Thus both $\mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}]$ and $\mathbf{S}^n(\mathbf{S}^k(V))$ have no one-dimensional irreducibles.

EXAMPLE 1. Let $V = V_0$, $k = 1$. Then $\mathbf{S}^n(\mathbf{S}^k(V)) = \mathrm{Sym}^n(V)$. It is known that $\mathbf{S}^n(V)$ is a $\mathrm{gl}(V)$-irreducible module. Hence it has an invariant only when $\dim(\mathrm{Sym}^n(V)) = 1$, which happens only when $\dim(V) = 1$ and $\mathrm{Sym}^n(V) = \mathbb{K}e_1^n$. Let us derive the fact that $\mathrm{Sym}^n(V)$ is $\mathrm{gl}(V)$-irreducible in our setting. Consider the symbolic operator

$$\bar{\mathbf{U}}\colon \mathrm{Super}^{[n]}[\bar{\mathscr{L}}_1 \mid \bar{\mathscr{P}}_{\mathbf{x}^-}] \to \mathrm{Sym}^n(V).$$

First we have that $\bar{\mathbf{U}}((\boldsymbol{\alpha}_1 \mid \mathbf{x})(\boldsymbol{\alpha}_2 \mid \mathbf{x}) \cdots (\boldsymbol{\alpha}_n \mid \mathbf{x})(\mathbf{S} \mid \mathbf{Der}_-(\lambda))) = 0$, if $\lambda_1 \geqslant 2$; to see this, let σ be the transposition switching a pair of letters in the first row of S. Then

$$\begin{aligned} &\sigma \cdot (\boldsymbol{\alpha}_1 \mid \mathbf{x})(\boldsymbol{\alpha}_2 \mid \mathbf{x}) \cdots (\boldsymbol{\alpha}_n \mid \mathbf{x})(\mathbf{S} \mid \mathbf{Der}_-(\lambda)) \\ &\quad = -(\boldsymbol{\alpha}_1 \mid \mathbf{x})(\boldsymbol{\alpha}_2 \mid \mathbf{x}) \cdots (\boldsymbol{\alpha}_n \mid \mathbf{x})(\mathbf{S} \mid \mathbf{Der}_-(\lambda)) \end{aligned}$$

in $\mathrm{Super}^{[n]}[\bar{\mathscr{L}} \mid \bar{\mathscr{P}}_{\mathbf{x}^-}]$. On the other hand $\bar{\mathbf{U}}(\sigma \cdot \mathbf{m}) = \bar{\mathbf{U}}(\mathbf{m})$ for all $\mathbf{m}$. So

$$\bar{\mathbf{U}}((\boldsymbol{\alpha}_1 \mid \mathbf{x})(\boldsymbol{\alpha}_2 \mid \mathbf{x}) \cdots (\boldsymbol{\alpha}_n \mid \mathbf{x})(\mathbf{S} \mid \mathbf{Der}_-(\lambda))) = 0.$$

Therefore, in order to have a non-zero image under $\bar{\mathbf{U}}$, the shape λ has to be $(1, 1, ..., 1) \vdash n$. There is only one standard tableau on $\bar{\mathscr{L}}_1$ of this shape, which is

$$\mathbf{S} = \begin{pmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \\ \cdots \\ \boldsymbol{\alpha}_n \end{pmatrix}.$$

Hence

$$\mathrm{Sym}^n(V) = \left\langle \bar{\mathbf{U}} \cdot (\boldsymbol{\alpha}_1 \mid \mathbf{x})(\boldsymbol{\alpha}_2 \mid \mathbf{x}) \cdots (\boldsymbol{\alpha}_n \mid \mathbf{x}) \left(\begin{array}{c|c} \boldsymbol{\alpha}_1 & \mathbf{e}_1 \\ \boldsymbol{\alpha}_2 & \mathbf{e}_1 \\ \cdots & \cdots \\ \boldsymbol{\alpha}_n & \mathbf{e}_1 \end{array} \right) \right\rangle_{\mathrm{gl}(V)} = \langle e_1^n \rangle_{\mathrm{gl}(V)},$$

which is irreducible.

EXAMPLE 2. Let $V = V_1$, $k = 1$. It can be shown similarly that $\mathbf{S}^n(\mathbf{S}^1(V)) = \Lambda^n(V)$ is $\mathrm{gl}(V)$-irreducible, and it has an invariant only when $n = d$. This invariant is $e_1 e_2 \cdots e_d$.

EXAMPLE 3. Let $k = 1$ and $V = V_0 \oplus V_1$ with $d_0 \neq 0$ and $d_1 \neq 0$. Let $e_i'^+ \in V_0$ and $e_i^- \in V_1$. We can argue similarly that

$$\mathbf{S}^n(\mathbf{S}^1(V)) = \mathbf{S}^n(V) = \langle [e_i']^n \rangle_{\mathrm{pl}(V)},$$

$$\Lambda^n(\mathbf{S}^1(V)) = \Lambda^n(V) = \langle [e_i]^n \rangle_{\mathrm{pl}(V)}.$$

Both of them are $\mathrm{pl}(V)$-irreducible. However, both of them are *not* irreducible when considered as $\mathrm{gl}(V_0) \oplus \mathrm{gl}(V_1)$-modules (recall that $\mathrm{gl}(V_0) \oplus \mathrm{gl}(V_1)$ is a *proper* Lie subalgebra of $\mathrm{pl}(V)$).

EXAMPLE 4. Let $V = V_1$ and $k = 2$. Consider the invariants of $\mathbf{S}^n(\mathbf{S}^2(V)) = \mathrm{Sym}^n(\Lambda^2(V))$, which is the case of classical Pfaffians. Use the symbolic operator

$$\mathrm{U}\colon \mathrm{Super}^{[n]}[\mathscr{L}_2 \mid \mathscr{P}_{x^+}] \to \mathrm{Sym}^n(\Lambda^2(V)).$$

The invariants of $\mathrm{Sym}^n(\Lambda^2(V))$ are spanned by $\mathrm{U}(\mathbf{m})$, where $\mathbf{m}$ is of the form

$$\mathbf{m} = (\alpha_1 \mid x)(\alpha_2 \mid x) \cdots (\alpha_n \mid x)(R_\lambda^+ \mid \mathrm{Der}_-(\lambda)),$$

for some even λ of the form $(d, d, ..., d)$, according to the first fundamental theorem and the regularization algorithm. An immediate consequence is that d has to be even and n has to be a multiple of $d/2$, in order to have non-trivial invariants in $\mathrm{Sym}^n(\Lambda^2(V))$. Moreover $\mathrm{U}(\mathbf{m})$ is a power of the Pfaffian,

$$\begin{aligned}&\mathrm{U}((\alpha_1 \mid x)(\alpha_2 \mid x) \cdots (\alpha_{d/2} \mid x)(\alpha_1\alpha_1\alpha_2\alpha_2 \cdots \alpha_{d/2}\alpha_{d/2} \mid e_1 e_2 \cdots e_d))\\ &\quad = d!\, 2^d \mathrm{Pfaffian}(M),\end{aligned}$$

where M is the skew-symmetric matrix $([e_i e_j]_{1 \leqslant i,j \leqslant d})$. Hence invariants of $\mathrm{Sym}(\Lambda^2(V))$ are polynomials in $\mathrm{Pfaffian}(M)$, as asserted in the classical first fundamental theorem for Pfaffians.

EXAMPLE 5. Let $V = V_0$ and $k = 2$. Consider the invariants of $\mathbf{S}^n(\mathbf{S}^2(V)) = \mathrm{Sym}^n(\mathrm{Sym}^2(V))$ and the symbolic operator

$$\bar{\mathrm{U}}\colon \mathrm{Super}^{[n]}[\bar{\mathscr{L}}_2 \mid \bar{\mathscr{P}}_{\mathbf{x}^-}] \to \mathrm{Sym}^n(\mathrm{Sym}^2(V)).$$

The invariants of $\mathrm{Sym}^n(\mathrm{Sym}^2(V))$ are spanned by $\bar{\mathrm{U}}(\mathbf{m})$, where $\mathbf{m}$ is of the form

$$\mathbf{m} = (\boldsymbol{\alpha}_1 \mid \mathbf{x})(\boldsymbol{\alpha}_2 \mid \mathbf{x}) \cdots (\boldsymbol{\alpha}_n \mid \mathbf{x})(\mathbf{R}_\lambda^- \mid \mathbf{Der}_-(\lambda)),$$

for some λ of the form $(d, d, ..., d) \vdash 2n$ and $\tilde{\lambda}$ even. Hence $d \mid n$ and $\bar{\mathrm{U}}(\mathbf{m})$ is a power of

$$\begin{aligned}&\bar{\mathrm{U}}\left((\boldsymbol{\alpha}_1 \mid \mathbf{x})(\boldsymbol{\alpha}_2 \mid \mathbf{x}) \cdots (\boldsymbol{\alpha}_d \mid \mathbf{x}) \left(\begin{matrix} \boldsymbol{\alpha}_1 & \boldsymbol{\alpha}_2 \cdots \boldsymbol{\alpha}_d \\ \boldsymbol{\alpha}_1 & \boldsymbol{\alpha}_2 \cdots \boldsymbol{\alpha}_d \end{matrix} \middle| \begin{matrix} \mathbf{e}_1 & \mathbf{e}_2 \cdots \mathbf{e}_d \\ \mathbf{e}_1 & \mathbf{e}_2 \cdots \mathbf{e}_d \end{matrix}\right)\right)\\ &\quad = (-1)^d\, d!\, \det(M),\end{aligned}$$

where M is the symmetric matrix $([e_ie_j]_{1 \leqslant i,j \leqslant d})$. Hence invariants of $\mathrm{Sym}^n(\mathrm{Sym}^2(V))$ are polynomials in $\det(M)$, which is the classical first fundamental theorem of invariants for Gramians.

EXAMPLE 6. Let $V = V_0$, $k = 2$. Consider the invariants of $\Lambda^n(\mathbf{S}^2(V)) = \Lambda^n(\mathrm{Sym}^2(V))$ and the symbolic operator

$$\bar{\mathbf{U}}' \colon \mathrm{Super}^{[n]}[\bar{\mathscr{L}}_2 \mid \bar{\mathscr{P}}_{\mathbf{x}^+}] \to \Lambda^n(\mathrm{Sym}^2(V)).$$

The invariants of $\Lambda^n(\mathrm{Sym}^2(V))$ are spanned by $\bar{\mathbf{U}}'(\mathbf{m})$, where $\mathbf{m}$ is of the form

$$\mathbf{m} = (\boldsymbol{\alpha}_1 \mid \mathbf{x})(\boldsymbol{\alpha}_2 \mid \mathbf{x}) \cdots (\boldsymbol{\alpha}_n \mid \mathbf{x})(\mathbf{F}_{\tilde{\lambda}}^- \mid \mathbf{Der}_-(\lambda)),$$

for some λ of the form $(d, d, ..., d) \vdash 2n$ and $\tilde{\lambda}$ of Frobenius type. Such a shape exists only when

$$\lambda = \underbrace{(d, d, ..., d)}_{d+1}$$

and so $d(d+1) = 2n$. Therefore $\Lambda^n(\mathrm{Sym}^2(V))$ has an invariant only when $n = d(d+1)/2$. From earlier discussion, such an invariant is given by

$$\mathscr{F}_{\tilde{\lambda}}^- = \prod_{1 \leqslant i \leqslant j \leqslant d} [e_ie_j].$$

EXAMPLE 7. Let $V = V_1$, $k = 2$. Consider the invariants of $\Lambda^n(\mathbf{S}^2(V)) = \Lambda^n(\Lambda^2(V))$ and the symbolic operator

$$\mathbf{U}' \colon \mathrm{Super}^{[n]}[\mathscr{L}_2 \mid \mathscr{P}_{x^-}] \to \Lambda^n(\Lambda^2(V)).$$

By an argument similar to the one of the preceding example, one can show that $\Lambda^n(\Lambda^2(V))$ admits a $\mathrm{gl}(V)$-invariant only when $n = d(d-1)/2$, and such an invariant is given by

$$\mathscr{F}_{\tilde{\lambda}}^+ = \prod_{1 \leqslant i < j \leqslant d} [e_ie_j],$$

where

$$\lambda = \underbrace{(d, d, ..., d)}_{d-1}.$$

9. THE SECOND FUNDAMENTAL THEOREM

We shall generalize the classical second fundamental theorem of the theory of invariants to the plethystic algebras $\mathbf{S}^n(\mathbf{S}^k(V))$, $\mathbf{S}^n(\mathbf{S}^k(V)^*)$,

$\Lambda^n(\mathbf{S}^k(V))$ and $\Lambda^n(\mathbf{S}^k(V)^*)$. Fix n and k and assume again that both $\dim(V_0)$ and $\dim(V_1)$ are large enough. As mentioned earlier

$$\mathrm{Super}^{[n]}[\bar{\mathscr{L}}_k \mid \bar{\mathscr{P}}_{\mathbf{x}}] = \bigoplus_{\lambda \vdash kn} \bigoplus_{\substack{\mathbf{S} \text{ standard on } \bar{\mathscr{L}}_k \\ \mathrm{sh}(\mathbf{S}) = \lambda}} (\mathbf{a}_1 \mid \mathbf{x})(\mathbf{a}_2 \mid \mathbf{x}) \cdots (\mathbf{a}_n \mid \mathbf{x})\, \mathfrak{W}_{\lambda \mathbf{S}},$$

where

$$\mathfrak{W}_{\lambda \mathbf{S}} = \langle (\mathbf{S} \mid \mathbf{Der}_-(\lambda)) \rangle_{\mathrm{pl}(V)}.$$

Define the *generalized Specht module of the first kind with multiplicity k* to be

$$\mathrm{Specht}_k^-(\lambda) = \langle (\mathbf{S} \mid \mathbf{Der}_-(\lambda)); \mathbf{S} \text{ is standard on } \bar{\mathscr{L}}_k, \mathrm{sh}(\mathbf{S}) = \lambda \rangle.$$

Similarly, we have

$$\mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}_x] = \bigoplus_{\lambda \vdash kn} \bigoplus_{\substack{S \text{ standard on } \mathscr{L}_k \\ \mathrm{sh}(S) = \lambda}} (\alpha_1 \mid x)(\alpha_2 \mid x) \cdots (\alpha_n \mid x)\, \mathfrak{S}_{\lambda S}.$$

Define the *generalized Specht module of the second kind with multiplicity k* to be

$$\mathrm{Specht}_k^+(\lambda) = \langle (S \mid \mathrm{Der}_-(\lambda)); S \text{ is standard on } \mathscr{L}_k, \mathrm{sh}(S) = \lambda \rangle.$$

Indeed, when $k = 1$, $\lambda \vdash n$, $\mathrm{Specht}_1^{\pm}(\lambda)$ are the classical Specht modules of the first kind and the second kind, both of which are S_n-irreducibles; see [14]. Clearly $\mathrm{Specht}_k^{\pm}(\lambda)$ are S_n-modules, where the actions are given by permutations of the letters α_i. In general, they are not S_n-irreducibles. The following discussion is confined to the symbolic operator $\mathbf{U}$, although the theory is completely similar for $\mathbf{U}'$, $\bar{\mathbf{U}}$, $\bar{\mathbf{U}}'$ and $\mathscr{U}$, $\mathscr{U}'$, $\bar{\mathscr{U}}$, $\bar{\mathscr{U}}'$.

First, as a vector space, $\mathrm{Specht}_k^+(\lambda)$ is a subspace of $\bigoplus_{S \text{ standard on } \mathscr{L}_k} \mathfrak{S}_{\lambda S}$. In fact

$$\langle \mathrm{Specht}_k^+(\lambda) \rangle_{\mathrm{pl}(V)} = \bigoplus_{S \text{ standard on } \mathscr{L}_k} \mathfrak{S}_{\lambda S}.$$

For convenience, let $X = (\alpha_1 \mid x)(\alpha_2 \mid x) \cdots (\alpha_n \mid x)$.

The first fundamental theorem discussed in the previous section can be rephrased as: any irreducible $\mathrm{pl}(V)$-module of $\mathbf{S}^n(\mathbf{S}^k(V))$ is of the form $\langle \mathbf{U}(X \cdot \mathbf{m}) \rangle_{\mathrm{pl}(V)}$, where $\mathbf{m}$ is an element in $\mathrm{Specht}_k^+(\lambda)$ for some λ.

Moreover, such a representative $\mathbf{m}$ is unique as stated in the following.

PROPOSITION 29. *If* $\mathbf{m}_1, \mathbf{m}_2 \in \mathrm{Specht}_k^+(\lambda)$, *and* $\langle \mathbf{m}_1 \rangle_{\mathrm{pl}(V)} = \langle \mathbf{m}_2 \rangle_{\mathrm{pl}(V)}$, *then* $\mathbf{m}_1 = c \cdot \mathbf{m}_2$ *for some* $c \in \mathbb{K}$.

Proof. Let $\mathbf{m}_1 = \sum_S c_S(S \mid \mathrm{Der}_-(\lambda))$ and $\mathbf{m}_2 = \sum_S d_S(S \mid \mathrm{Der}_-(\lambda))$. Then both the sets $\{\sum_S c_S(S \mid \boxed{T}\,);\ T \text{ is standard}\}$ and $\{\sum_S d_S(S \mid \boxed{T}\,);\ T \text{ is standard}\}$ are bases of the same $\mathrm{pl}(V)$-irreducible. In particular

$$\sum_S c_S(S \mid \mathrm{Der}_-(\lambda)) = \sum_T k_T \cdot \sum_S d_S(S \mid \boxed{T}\,).$$

Therefore $k_T = 0$ whenever $T \neq \mathrm{Der}_-(\lambda)$, since standard right symmetrized tableaux $(S \mid \boxed{T}\,)$ form a basis of $\mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}]$. Thus $\mathbf{m}_1 = k_{\mathrm{Der}_-(\lambda)} \cdot \mathbf{m}_2$ as desired. ∎

This proposition is the analog of the classical fact that the highest vector is unique (up to a scalar) in an irreducible $\mathrm{sl}(V)$-module.

PROPOSITION 30. *Let $\mathbf{m}_1, \mathbf{m}_2, ..., \mathbf{m}_h \in \mathrm{Specht}_k^+(\lambda)$. Then they are linearly independent if and only if the sum*

$$\langle \mathbf{m}_1 \rangle_{\mathrm{pl}(V)} + \langle \mathbf{m}_2 \rangle_{\mathrm{pl}(V)} + \cdots + \langle \mathbf{m}_h \rangle_{\mathrm{pl}(V)} \tag{44}$$

is direct.

Proof. If the sum is direct, then

$$c_1 \cdot \mathbf{m}_1 + c_2 \cdot \mathbf{m}_2 + \cdots + c_h \cdot \mathbf{m}_h = 0$$

implies $c_1 = c_2 = \cdots = c_h = 0$, so they are linearly independent.

Conversely, if $\mathbf{m}_1, \mathbf{m}_2, ..., \mathbf{m}_h$ are linearly independent, let

$$\mathbf{m}_i = \sum_{S \text{ standard on } \mathscr{L}_k} c_{iS}(S \mid \mathrm{Der}_-(\lambda)).$$

Then the set

$$\bigcup_i \left\{ \sum_S c_{iS}(S \mid \boxed{T}\,) \colon T \text{ is standard on } \mathscr{P} \right\} \tag{45}$$

spans the vector space $\sum_i \langle \mathbf{m}_i \rangle_{\mathrm{pl}(V)}$. The sum in (44) is direct if and only if the set in (45) is linearly independent. To see that this is indeed the case, let

$$\sum_i \sum_T k_{iT} \cdot \sum_S c_{iS}(S \mid \boxed{T}\,) = 0.$$

Then $\sum_i k_{iT} \cdot c_{iS} = 0$ for all T and S, since the right symmetrized tableaux $(S \mid \boxed{T}\,)$ are a basis of $\mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}]$. Note that the system

$$\sum x_i \cdot c_{iS} = 0 \qquad \text{for all } S$$

has only $\mathbf{0}$ as solution, since $\mathbf{m}_i$ are linearly independent. Therefore $k_{iT}=0$ for all i and T, as desired. ∎

The letterplace algebra $\mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}_x]$ is both a $\mathrm{pl}(V)$-module by right polarizations and an S_n-module by permuting the letters. Clearly, we have

PROPOSITION 31. *The two actions of* $\mathrm{pl}(V)$ *and* S_n *on* $\mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}_x]$ *commute.*

It follows that the space of S_n-invariants

$$\mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}_x]^{S_n} \stackrel{\text{def}}{=} \{\mathbf{m} \in \mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}_x];\, \sigma \cdot \mathbf{m} = \mathbf{m} \text{ for all } \sigma \in S_n\}$$

is a $\mathrm{pl}(V)$-module.

THEOREM 11 (The Second Fundamental Theorem). (i) *The symbolic operator* $\mathbf{U}$: $\mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}_x]^{S_n} \to \mathbf{S}^n(\mathbf{S}^k(V))$ *is a* $\mathrm{pl}(V)$*-isomorphism.*

(ii) *Let* $\mathbf{m} \in \mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}_x]$. *Then* $\mathbf{m} \in \ker(\mathbf{U})$ *if and only if* $\sum_{\sigma \in S_n} \sigma \cdot \mathbf{m} = \mathbf{0}$ *in* $\mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}_x]$.

Proof. (i) We know that $\mathbf{U}$ is $\mathrm{pl}(V)$-equivariant. On the other hand, when restricted on $\mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}_x]^{S_n}$, the inverse of $\mathbf{U}$ can be given by

$$\begin{aligned}
&\mathbf{U}^{-1}([e_{i_1}e_{i_2}\cdots e_{i_k}][e_{j_1}e_{j_2}\cdots e_{j_k}]\cdots[e_{h_1}e_{h_2}\cdots e_{h_k}]) \\
&\quad = \frac{1}{n!}\sum_{\sigma \in S_n} (\alpha_{\sigma(1)}^{(k+1)} \mid x e_{i_1}e_{i_2}\cdots e_{i_k})(\alpha_{\sigma(2)}^{(k+1)} \mid x e_{j_1}e_{j_2}\cdots e_{j_k}) \\
&\qquad \cdots(\alpha_{\sigma(n)}^{(k+1)} \mid x e_{h_1}e_{h_2}\cdots e_{h_k}).
\end{aligned}$$

Therefore $\mathbf{U}$ is a $\mathrm{pl}(V)$-isomorphism when restricted on $\mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}_x]^{S_n}$.

(ii) The assertion is immediate from (i) and from the fact that

$$\mathbf{U}(\mathbf{m}) = 0 \Leftrightarrow \mathbf{U}\left(\sum_{\sigma \in S_n} \sigma \cdot \mathbf{m}\right) = 0. \quad ∎$$

The following theorem provides a more explicit structure for $\mathbf{S}^n(\mathbf{S}^k(V))$.

THEOREM 12. *Let* $\mathbf{S}^n(\mathbf{S}^k(V))_\lambda$ *denote the isotypic component corresponding to the Schur modules of shape* λ. *Then*

$$\mathbf{S}^n(\mathbf{S}^k(V))_\lambda = \langle \mathbf{U} \cdot (X \cdot \mathrm{Specht}_k^+(\lambda))^{S_n} \rangle_{\mathrm{pl}(V)}. \tag{46}$$

Furthermore, $\mathit{length}_{\mathrm{pl}(V)}(\mathbf{S}^n(\mathbf{S}^k(V))_\lambda)$, *which is the number of irreducibles occurring in a decomposition of* $\mathbf{S}^n(\mathbf{S}^k(V))_\lambda$ *into* $\mathrm{pl}(V)$*-irreducibles, equals* $\dim_{\mathbb{K}}((X\cdot \mathrm{Specht}_k^+(\lambda))^{S_n}$.

Proof. By the previous theorem,

$$\mathbf{S}^n(\mathbf{S}^k(V))_\lambda = \mathrm{U}((\mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}_x]^{S_n})_\lambda),$$

where $(\mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}_x]^{S_n})_\lambda$ denotes the isotypic component of $\mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}_x]^{S_n}$. We claim

$$(\mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}_x]^{S_n})_\lambda = (\mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}_x]_\lambda)^{S_n}, \tag{47}$$

where

$$\mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}_x]_\lambda = \bigoplus_{\substack{S \text{ standard on } \mathscr{L}_k\\ \mathrm{sh}(S)=\lambda}} X\cdot \mathfrak{S}_{\lambda S}.$$

In fact, setting $\Gamma_n = \sum_{\sigma\in S_n}\sigma$, we have

$$\begin{aligned}(\mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}_x])^{S_n} &= \Gamma_n\cdot(\mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}_x])\\ &= \Gamma_n\cdot\left(\bigoplus_\lambda \mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}_x]_\lambda\right)\\ &= \bigoplus_\lambda \Gamma_n\cdot(\mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}_x]_\lambda);\end{aligned} \tag{48}$$

the last sum above being direct follows from the fact that

$$\Gamma_n\cdot(\mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}_x]_\lambda) \subseteq \mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}_x]_\lambda.$$

Since each $\mathrm{pl}(V)$-module $\mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}_x]_\lambda$ is semisimple and consists of only mutually isomorphic $\mathrm{pl}(V)$-irreducibles, the same is true for its submodule

$$\Gamma_n\cdot(\mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}_x]_\lambda).$$

Furthermore, since any $\mathrm{pl}(V)$-irreducible in $\mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}_x]_\lambda$ is not $\mathrm{pl}(V)$-isomorphic to any $\mathrm{pl}(V)$-irreducible in $\mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}_x]_\mu$, $\lambda\neq\mu$, we know that any $\mathrm{pl}(V)$-irreducible in $\Gamma_n\cdot(\mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}_x]_\lambda)$ is not $\mathrm{pl}(V)$-isomorphic to any $\mathrm{pl}(V)$-irreducible in $\Gamma_n\cdot(\mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}_x]_\mu)$. Hence, formula (48) gives the decomposition of $(\mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}_x])^{S_n}$ into isotypic components. Thus

$$\begin{aligned}\Gamma_n\cdot(\mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}_x]_\lambda) &= (\mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}_x]_\lambda)^{S_n}\\ &= (\mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}_x]^{S_n})_\lambda.\end{aligned}$$

Now

$$(\mathrm{Super}^{[n]}[\mathscr{L}_k \mid \mathscr{P}_x]_\lambda)^{S_n} = (\langle X \cdot \mathrm{Specht}_k^+(\lambda) \rangle_{\mathrm{pl}(V)})^{S_n}$$
$$= \Gamma_n(\langle X \cdot \mathrm{Specht}_k^+(\lambda) \rangle_{\mathrm{pl}(V)}).$$

The last expression is equal to

$$\langle \Gamma_n(X \cdot \mathrm{Specht}_k^+(\lambda)) \rangle_{\mathrm{pl}(V)} = \langle (X \cdot \mathrm{Specht}_k^+(\lambda))^{S_n} \rangle_{\mathrm{pl}(V)}.$$

Hence (46) follows when **U** is applied. The second statement in the theorem follows from Proposition 30. ∎

Let $\mathfrak{Schur}_\lambda$ denote the abstract pl(V)-irreducible isomorphic to $\mathfrak{S}_{\lambda S}$. As a corollary, we have

COROLLARY 32.

$$\mathbf{S}^n(\mathbf{S}^k(V)) \cong \bigoplus_{\lambda \vdash kn} \dim_{\mathbb{K}}((X \cdot \mathrm{Specht}_k^+(\lambda))^{S_n})\, \mathfrak{Schur}_\lambda. \tag{49}$$

Translating our earlier results about $\mathbf{S}^n(\mathbf{S}^2(V))$ and $\Lambda^n(\mathbf{S}^2(V))$, we get

$$\dim_{\mathbb{K}}(((\alpha_1^+ \mid x^+) \cdots (\alpha_n^+ \mid x^+) \cdot \mathrm{Specht}_2^+(\lambda))^{S_n}) = \begin{cases} 1 & \text{if } \lambda \text{ is even} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\dim_{\mathbb{K}}(((\alpha_1^+ \mid x^-) \cdots (\alpha_n^+ \mid x^-) \cdot \mathrm{Specht}_2^+(\lambda))^{S_n})$$
$$= \begin{cases} 1 & \text{if } \lambda \text{ is of Frobenius type} \\ 0 & \text{otherwise.} \end{cases}$$

To end the discussion, we make a final remark about the assumption of $d_0 = \dim(V_0)$ and $d_1 = \dim(V_1)$ being large enough. First of all, the number $\dim_{\mathbb{K}}((\mathrm{Specht}_k^+(\lambda))^{S_n})$ does not depend on d_0 and d_1, as long as $d_1 \geqslant \lambda_1$. If $d_1 < \lambda_1$ and if $\lambda_{d_0+1} \leqslant d_1$, then there exists at least one standard tableau T on $\mathscr{P}$. If we modify the notion of Specht module by defining

$$\mathrm{Specht}_k^+(T) = \langle (S \mid \boxed{T}\,);\ S \text{ is standard on } \mathscr{L}_k,\ \mathrm{sh}(S) = \lambda \rangle_{\mathbb{K}}$$

then all the previous discussions still work. In particular, we will get the formula

$$\mathbf{S}^n(\mathbf{S}^k(V)) = \bigoplus_{\lambda \vdash kn} \dim_{\mathbb{K}}((X \cdot \mathrm{Specht}_k^+(T))^{S_n})\, \mathfrak{Schur}_\lambda.$$

Clearly

$$\dim_{\mathbb{K}}((X \cdot \mathrm{Specht}_k^+(\lambda))^{S_n}) = \dim_{\mathbb{K}}((X \cdot \mathrm{Specht}_k^+(T))^{S_n}).$$

Thus, the structure of $\mathbf{S}^n(\mathbf{S}^k(V))$ as a pl(V)-module will not change at all by assuming that d_1 is big enough, except that the shapes λ not satisfying the condition $\lambda_{d_0+1} \leqslant d_1$ will simply disappear in a complete decomposition of a plethystic algebra into pl(V)-irreducibles. Similarly, the assumption of d_0 being large enough when we use the Weyl modules will not change the structure of the plethystic algebra either.

10. Symbolic–Umbral Operators and Weitzenböck's Method of "Complex Symbols"

The definition of the symbolic operators and of the umbral operators for the plethystic algebras $\mathbf{S}(\mathbf{S}(V))$, $\Lambda(\mathbf{S}(V))$, $\mathbf{S}(\mathbf{S}(V)^*)$ and $\Lambda(\mathbf{S}(V)^*)$ can be easily extended to the plethystic algebras with more than one component such as $\mathbf{S}(\mathbf{S}(V) \oplus \mathbf{S}(V) \oplus \cdots \oplus \mathbf{S}(V))$ and $\Lambda(\mathbf{S}(V) \oplus \mathbf{S}(V) \oplus \cdots \oplus \mathbf{S}(V))$, by enlarging the letter set $\mathscr{L}$ to a disjoint union of sets of letters. All the previous statements can be made true by suitable modifications. For example, let us consider the second fundamental theorem for $\mathbf{S}^n(\mathbf{S}^{k_1}(V) \oplus \mathbf{S}^{k_2}(V) \oplus \cdots \oplus \mathbf{S}^{k_p}(V))$. Let $\mathbf{S}^{n_1}(\mathbf{S}^{k_1}(V)) \otimes \mathbf{S}^{n_2}(\mathbf{S}^{k_2}(V)) \otimes \cdots \otimes \mathbf{S}^{n_p}(\mathbf{S}^{k_p}(V))$ be a pl(V)-submodule where $n_1 + n_2 + \cdots + n_p = n$. Set $\mathscr{L} = \mathscr{L}_1 \cup \mathscr{L}_2 \cup \cdots \cup \mathscr{L}_p$, where $\mathscr{L}_i$ is a set of letters associated to the ith component $\mathbf{S}^{k_i}(V)$, with $|\mathscr{L}_i| = n_i$. Denote by $\mathrm{Super}^{[\mathbf{n}]}[\mathscr{L}_{\mathbf{k}} \mid \mathscr{P}_x]$ the subspace of $\mathrm{Super}[\mathscr{L} \mid \mathscr{P}_x]$ spanned by the monomials

$$(\alpha_1^{(k_1+1)} \mid x\omega_1) \cdots (\alpha_{n_1}^{(k_1+1)} \mid x\omega_{n_1})(\alpha_1'^{(k_2+1)} \mid x\omega_1') \cdots (\alpha_{n_2}'^{(k_2+1)} \mid x\omega_{n_2}') \\ \cdots (\alpha_1''^{(k_p+1)} \mid x\omega_1'') \cdots (\alpha_{n_p}''^{(k_p+1)} \mid x\omega_{n_p}''), \qquad (50)$$

where $\alpha_i \in \mathscr{L}_1$, $\alpha_i' \in \mathscr{L}_2, ..., \alpha_i'' \in \mathscr{L}_p$, and ω_i, $\omega_i', ..., \omega_i''$ are monomials in $\mathrm{Super}[\mathscr{P}]$ of suitable lengths. The symbolic linear operator

$$\mathbf{U}\colon \mathrm{Super}^{[\mathbf{n}]}[\mathscr{L}_{\mathbf{k}} \mid \mathscr{P}_{x^+}] \to \mathbf{S}^{n_1}(\mathbf{S}^{k_1}(V)) \otimes \cdots \otimes \mathbf{S}^{n_p}(\mathbf{S}^{k_p}(V))$$

is defined such that the monomial in (50) is mapped to

$$[\omega_1] \cdots [\omega_{n_1}][\omega_1'] \cdots [\omega_{n_2}'] \cdots [\omega_1''] \cdots [\omega''_{n_p}].$$

Consider the obvious action of $S_{n_1} \times S_{n_2} \times \cdots \times S_{n_p}$ on $\mathrm{Super}^{[\mathbf{n}]}[\mathscr{L}_{\mathbf{k}} \mid \mathscr{P}_{x^+}]$ by permuting the letters; one can check that

$$\mathbf{U}((\sigma_1, \sigma_2, ..., \sigma_p) \cdot \mathbf{m}) = \mathbf{U}(\mathbf{m})$$

for all $\mathbf{m} \in \mathrm{Super}^{[\mathbf{n}]}[\mathscr{L}_{\mathbf{k}} \mid \mathscr{P}_{x^+}]$, $\sigma_i \in S_{n_i}$. The second fundamental theorem can be stated as follows. Let $\mathbf{m} \in \mathrm{Super}^{[\mathbf{n}]}[\mathscr{L}_{\mathbf{k}} \mid \mathscr{P}_{x^+}]$; then $\mathbf{U}(\mathbf{m}) = 0$ if and only if

$$\sum_{\sigma_i \in S_{n_i}} (\sigma_1, \sigma_2, ..., \sigma_p) \cdot \mathbf{m} = 0 \qquad \text{in} \quad \mathrm{Super}^{[\mathbf{n}]}[\mathscr{L}_{\mathbf{k}} \mid \mathscr{P}_{x^+}].$$

In this section, we are interested in the plethystic algebra $\mathrm{Sym}(\bigoplus_i W_i)$, where each W_i is $\mathrm{Sym}^{k_1}(V)$, or $\Lambda^{k_2}(V)$, or $\mathrm{Sym}^{h_1}(V)^*$, or $\Lambda^{h_2}(V)^*$. The $\mathrm{gl}(V)$-invariants of the $\mathrm{gl}(V)$-module $\mathrm{Sym}(\bigoplus_i W_i)$ are important examples of classical *concomitants*. To introduce the general symbolic–umbral method for $\mathrm{Sym}(\bigoplus_i W_i)$, we first define the symbolic–umbral operator for the special case $\mathrm{Sym}(\mathrm{Sym}(V)\oplus\Lambda(V)^*)$. To problem of finding an analog of such an operator for $\mathbf{S}(\mathbf{S}(V)\oplus\Lambda(V)^*)$ is still open if V is $\mathbb{Z}_2$-graded.

Consider the $\mathrm{gl}(V)$-submodule of $\mathrm{Sym}(\mathrm{Sym}(V)\oplus\Lambda(V)^*)$:

$$\mathrm{Sym}^{n_1}(\mathrm{Sym}^{k_1}(V))\otimes\mathrm{Sym}^{n_2}(\Lambda^{k_2}(V)^*).$$

Let $\mathscr{L}_1=\{\alpha_1^-,\alpha_2^-,...,\alpha_{n_1}^-\}$ and $\mathscr{L}_2=\{\beta_1^{*+},\ \beta_2^{*+},...,\beta_{n_2}^{*+}\}$ be two sets of letters. Write $\mathscr{L}=\mathscr{L}_1\cup\mathscr{L}_2$. Let $\mathscr{P}=\{e_1^-,e_2^-,...,e_d^-\}$ and $\mathscr{P}^*=(f_1^-,f_2^-,...,f_d^-\}$ be bases of V and V^*, dual to each other. Let x_1^-, x_2 be mock places, where $|x_2|=(-1)^{k_2}$. Write $\mathbb{P}=\mathscr{P}\cup\mathscr{P}^*$ and $\mathbb{P}_{\mathbf{x}}=\mathbb{P}\cup\{x_1,x_2\}$. Let $\mathrm{Super}^{[\mathbf{n}]}[\mathscr{L}_{\mathbf{k}}\mid\mathbb{P}_{\mathbf{x}}]$ be the $\mathrm{gl}(V)$-submodule of $\mathrm{Super}[\mathscr{L}\mid\mathbb{P}_{\mathbf{x}}]$ spanned by the monomials

$$\begin{aligned}&(\alpha_1\mid x_1)(\alpha_1\mid e_{i_1})\cdots(\alpha_1\mid e_{i_{k_1}})\cdots(\alpha_{n_1}\mid x_1)(\alpha_{n_1}\mid e_{j_1})\cdots(\alpha_{n_1}\mid e_{j_{k_1}})\\&\quad\times(\beta_1^{*(k_2+1)}\mid x_2f_{s_1}\cdots f_{s_{k_2}})\cdots(\beta_{n_2}^{*(k_2+1)}\mid x_2f_{t_1}\cdots f_{t_{k_2}}),\end{aligned}\tag{51}$$

where the action is given by

$$\begin{aligned}&E_{ij}\cdot(\alpha_s\mid e_t)=\ (\alpha_s\mid E_{ij}\cdot e_t)\ =\delta_{jt}(\alpha_s\mid e_i),\\&E_{ij}\cdot(\beta_s^*\mid f_t)=(\beta_s^*\mid E_{ij}\odot f_t)=-\delta_{it}(\beta_s^*\mid f_j),\\&E_{ij}\cdot(\alpha_s\mid x_1)=\ E_{ij}\cdot(\alpha_s\mid x_2)\ =E_{ij}\cdot(\beta_s^*\mid x_1)=E_{ij}\cdot(\beta_s^*\mid x_2)=0,\end{aligned}$$

and extended by (even) derivation.

Define the *first level symbolic–umbral operator*

$$\mathfrak{U}_1:\mathrm{Super}^{[\mathbf{n}]}[\mathscr{L}_{\mathbf{k}}\mid\mathbb{P}_{\mathbf{x}}]\to\mathrm{Sym}^{n_1}(\mathrm{Sym}^{k_1}(V))\otimes\mathrm{Sym}^{n_2}(\Lambda^{k_2}(V)^*),$$

such that the monomial in (51) is mapped to

$$[e_{i_1}e_{i_2}\cdots e_{i_{k_1}}]\cdots[e_{j_1}e_{j_2}\cdots e_{j_{k_1}}][f_{s_1}f_{s_2}\cdots f_{s_{k_2}}]\cdots[f_{t_1}f_{t_2}\cdots f_{t_{k_2}}].$$

It is easy to check the following proposition.

PROPOSITION 33. *The first level symbolic–umbral operator* $\mathfrak{U}_1$ *is well defined, surjective, and* $\mathrm{gl}(V)$*-equivariant.*

Since the action of $\mathrm{gl}(V)$ on $\mathrm{Super}^{[\mathbf{n}]}[\mathscr{L}_{\mathbf{k}}\mid\mathbb{P}_{\mathbf{x}}]$ is a mixture of cogradient action on $\mathscr{P}$ and contragradient action on $\mathscr{P}^*$, the structure of $\mathrm{Super}^{[\mathbf{n}]}[\mathscr{L}_{\mathbf{k}}\mid\mathbb{P}_{\mathbf{x}}]$ as a $\mathrm{gl}(V)$-module remains to be revealed. To study its structure, first let $\mathrm{Super}^{[\mathbf{n}]}[\mathscr{L}_{\mathbf{k}} \mid \mathbb{P}]$ be the $gl(V)$-submodule of $\mathrm{Super}[\mathscr{L} \mid \mathbb{P}]$ spanned by monomials of the form

$$(\alpha_1 \mid e_{i_1}) \cdots (\alpha_1 \mid e_{i_{k_1}}) \cdots (\alpha_{n_1} \mid e_{j_1}) \cdots (\alpha_{n_1} \mid e_{j_{k_1}}) \\ \times (\beta_1^{*(k_2)} \mid f_{s_1} \cdots f_{s_{k_2}}) \cdots (\beta_{n_2}^{*(k_2)} \mid f_{t_1} \cdots f_{t_{k_2}}).$$

Clearly, we have a $gl(V)$-isomorphism between the $gl(V)$-modules $\mathrm{Super}^{[\mathbf{n}]}[\mathscr{L}_{\mathbf{k}} \mid \mathbb{P}_{\mathbf{x}}]$ and $\mathrm{Super}^{[\mathbf{n}]}[\mathscr{L}_{\mathbf{k}} \mid \mathbb{P}]$ via the map

$$(\alpha_1 \mid x_1) \cdots (\alpha_{n_1} \mid x_1)(\beta_1^* \mid x_2) \cdots (\beta_{n_2}^* \mid x_2)\, \mathbf{m} \mapsto \mathbf{m},$$

for every element $\mathbf{m} \in \mathrm{Super}^{[\mathbf{n}]}[\mathscr{L}_{\mathbf{k}} \mid \mathbb{P}]$.

Next, associate with each letter $\beta_i^* \in \mathscr{L}_2$ a new set of letters $\{\beta_{i1}^+, \beta_{i2}^+, ..., \beta_{ik_2}^+\}$. Take the (disjoint) union

$$\mathbb{L} = \mathscr{L}_1 \,\dot\cup \left(\dot\bigcup_i \{\beta_{i1}^+, \beta_{i2}^+, ..., \beta_{ik_2}^+\}\right).$$

Let y be another mock place such that $|y| = (-1)^d$. Write $\mathscr{P}_y = \mathscr{P} \,\dot\cup\, y$. Let $\mathrm{Super}^{[\mathbf{n}]}[\mathbb{L}_{\mathbf{k}} \mid \mathscr{P}_y]$ be the $gl(V)$-submodule of $\mathrm{Super}[\mathbb{L} \mid \mathscr{P}_y]$ spanned by the monomials of type

$$(\alpha_1 \mid e_{i_1}) \cdots (\alpha_1 \mid e_{i_{k_1}}) \cdots (\alpha_{n_1} \mid e_{j_1}) \cdots (\alpha_{n_1} \mid e_{j_{k_1}})(\beta_{11}^{(d)} \mid y\omega_{11}) \\ \cdots (\beta_{1k_2}^{(d)} \mid y\omega_{1k_2}) \cdots (\beta_{n_21}^{(d)} \mid y\omega_{n_21}) \cdots (\beta_{n_2k_2}^{(d)} \mid y\omega_{n_2k_2}), \tag{52}$$

where ω_{ij} are monomials in $\mathrm{Super}[\mathscr{P}] = \Lambda(P)$ of length $d-1$, and where the action is the cogradient action of $gl(V)$ on $\mathscr{P}$.

Define the *second level symbolic–umbral operator*

$$\mathfrak{U}_2 : \mathrm{Super}^{[\mathbf{n}]}[\mathbb{L}_{\mathbf{k}} \mid \mathscr{P}_y] \to \mathrm{Super}^{[\mathbf{n}]}[\mathscr{L}_{\mathbf{k}} \mid \mathbb{P}],$$

such that the monomial in (52) is mapped to

$$(\alpha_1 \mid e_{i_1}) \cdots (\alpha_1 \mid e_{i_{k_1}}) \cdots (\alpha_{n_1} \mid e_{j_1}) \cdots (\alpha_{n_1} \mid e_{j_{k_1}})(-1)^{d-s_1} (\beta_1^* \mid f_{s_1}) \\ \cdots (-1)^{d-s_{k_2}} (\beta_1^* \mid f_{s_{k_2}}) \cdots (-1)^{d-t_1} (\beta_{n_2}^* \mid f_{t_1}) \cdots (-1)^{d-t_{k_2}} (\beta_{n_2}^* \mid f_{t_{k_2}}),$$

where we assume

$$\omega_{11} = e_1 \cdots \hat{e}_{s_1} \cdots e_d, ..., \quad \omega_{1k_2} = e_1 \cdots \hat{e}_{s_{k_2}} \cdots e_d,$$

$$\cdots$$

$$\omega_{n_21} = e_1 \cdots \hat{e}_{t_1} \cdots e_d, ..., \quad \omega_{n_2k_2} = e_1 \cdots \hat{e}_{t_{k_2}} \cdots e_d.$$

In other words, $\mathfrak{U}_2$ is the "algebraic" map such that

$$(\alpha_i \mid e_j) \mapsto (\alpha_i \mid e_j), \\ (\beta_{ih}^{(d)} \mid ye_1 \cdots \hat{e}_j \cdots e_d) \mapsto (-1)^{d-j} (\beta_i^* \mid f_j).$$

The extension of the operator $\mathfrak{U}_2$ to the operator from $\bigoplus_{\mathbf{n},\mathbf{k}} \mathrm{Super}^{[\mathbf{n}]}[\mathbb{L}_{\mathbf{k}} \mid \mathscr{P}_y]$ to $\bigoplus_{\mathbf{n},\mathbf{k}} \mathrm{Super}^{[\mathbf{n}]}[\mathscr{L}_{\mathbf{k}} \mid \mathbb{P}]$ will still be denoted as $\mathfrak{U}_2$. The linear operator $\mathfrak{U}_2$ also possesses the same multiplicative property as the symbolic operator $\mathbf{U}$ (for details, see the last paragraph in Section 2).

PROPOSITION 34. *The second level symbolic–umbral operator* $\mathfrak{U}_2$ *is well defined, surjective, and* $\mathrm{sl}(V)$*-equivariant.*

Proof. The well-definedness follows from the use of the mock place y. To see that $\mathfrak{U}_2$ is $\mathrm{sl}(V)$-equivariant, it is sufficient to check that

$$\begin{aligned}\mathfrak{U}_2(E_{st}\cdot(\beta_{ih}^{(d)} \mid ye_1\cdots\hat{e}_j\cdots e_d)) &= (-1)^{d-j} E_{st}\cdot(\beta_i^* \mid f_j)\\ &= -(-1)^{d-j}\delta_{js}(\beta_i^* \mid f_t), \qquad s\neq t, \qquad (53)\end{aligned}$$

and

$$\begin{aligned}&\mathfrak{U}_2((E_{ss}-E_{s+1\,s+1})\cdot(\beta_{ih}^{(d)} \mid ye_1\cdots\hat{e}_j\cdots e_d))\\ &\quad=(-1)^{d-j}(E_{ss}-E_{s+1\,s+1})\cdot(\beta_i^* \mid f_j)\\ &\quad=(-1)^{d-j}(\delta_{j\,s+1}-\delta_{js})(\beta_i^* \mid f_j). \qquad (54)\end{aligned}$$

To prove (53), note that the left side equals

$$\begin{aligned}&(1-\delta_{tj})\,\mathfrak{U}_2(E_{st}\cdot(\beta_{ih}^{(d)} \mid ye_1\cdots\hat{e}_j\cdots e_d))\\ &\quad=(1-\delta_{tj})\,\delta_{sj}(-1)^{j-t-1}\,\mathfrak{U}_2\cdot(\beta_{ih}^{(d)} \mid ye_1\cdots\hat{e}_t\cdots e_d)\\ &\quad=(1-\delta_{tj})\,\delta_{sj}(-1)^{d+j-1}(\beta_i^* \mid f_t)\\ &\quad=-(-1)^{d-j}\delta_{sj}(\beta_i^* \mid f_t),\end{aligned}$$

where the last step follows from the assumption $s\neq t$.

To prove (54), note that the left side equals

$$\begin{aligned}&[(1-\delta_{sj})-(1-\delta_{s+1\,j})]\,\mathfrak{U}_2\cdot(\beta_{ih}^{(d)} \mid ye_1\cdots\hat{e}_j\cdots e_d)\\ &\quad=(-1)^{d-j}(\delta_{s+1\,j}-\delta_{sj})(\beta_i^* \mid f_j). \quad\blacksquare\end{aligned}$$

We note that the operator $\mathfrak{U}_2$ is not $\mathrm{gl}(V)$-equivariant. However, since

$$\mathrm{gl}(V)=\mathrm{sl}(V)\oplus\langle E_{11}+E_{22}+\cdots+E_{dd}\rangle_{\mathbb{K}}$$

and the action of $E_{11}+E_{22}+\cdots+E_{dd}$ on $\mathrm{Super}^{[\mathbf{n}]}[\mathbb{L}_{\mathbf{k}} \mid \mathscr{P}_y]$ or on $\mathrm{Super}^{[\mathbf{n}]}[\mathscr{L}_{\mathbf{k}} \mid \mathbb{P}]$ is a scalar multiplication when restricted to elements of homogeneous contents in $\mathscr{P}$ or in $\mathbb{P}$, any $\mathrm{gl}(V)$-irreducible of $\mathrm{Super}^{[\mathbf{n}]}[\mathbb{L}_{\mathbf{k}} \mid \mathscr{P}_y]$ or $\mathrm{Super}^{[\mathbf{n}]}[\mathscr{L}_{\mathbf{k}} \mid \mathbb{P}]$ is an $\mathrm{sl}(V)$-irreducible and vice versa. Therefore, although $\mathfrak{U}_2$ is only $\mathrm{sl}(V)$-equivariant, any $\mathrm{gl}(V)$-irreducible of $\mathrm{Super}^{[\mathbf{n}]}[\mathbb{L}_{\mathbf{k}} \mid \mathscr{P}_y]$ is still mapped either to an $\mathrm{sl}(V)$-irreducible of $\mathrm{Super}^{[\mathbf{n}]}[\mathscr{L}_{\mathbf{k}} \mid \mathbb{P}]$ or to zero.

In particular, let us study the sl(V)-invariants of $\mathrm{Super}^{[\mathbf{n}]}[\mathscr{L}_{\mathbf{k}} \mid \mathbb{P}]$ next. We know that they form the isotypic component consisting of one-dimensional irreducibles. Let $\mathrm{Super}^{[\mathbf{n}]}[\mathbb{L}_{\mathbf{k}} \mid \mathscr{P}]$ be the gl(V)-submodules of $\mathrm{Super}[\mathbb{L} \mid \mathscr{P}]$ spanned by the monomials of the form

$$(\alpha_1 \mid e_{i_1}) \cdots (\alpha_1 \mid e_{i_{k_1}}) \cdots (\alpha_{n_1} \mid e_{j_1}) \cdots (\alpha_{n_1} \mid e_{j_{k_1}})$$
$$\times (\beta_{11}^{(d-1)} \mid \omega_{11}) \cdots (\beta_{1k_2}^{(d-1)} \mid \omega_{1k_2}) \cdots (\beta_{n_2 1}^{(d-1)} \mid \omega_{n_2 1}) \cdots (\beta_{n_2 k_2}^{(d-1)} \mid \omega_{n_2 k_2}),$$

where ω_{ij} are monomials in $\mathrm{Super}[\mathscr{P}]$ of length $d-1$. Clearly, the gl(V)-modules $\mathrm{Super}^{[\mathbf{n}]}[\mathbb{L}_{\mathbf{k}} \mid \mathscr{P}_y]$ and $\mathrm{Super}^{[\mathbf{n}]}[\mathbb{L}_{\mathbf{k}} \mid \mathscr{P}]$ are gl(V)-isomorphic.

PROPOSITION 35 (Regularization Algorithm). *Every* sl(V)-*invariant of* $\bigoplus_{\mathbf{n},\mathbf{k}} \mathrm{Super}^{[\mathbf{n}]}[\mathbb{L}_{\mathbf{k}} \mid \mathscr{P}]$ *is a polynomial in elements of the forms*

$$(\alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_d} \mid e_1 e_2 \cdots e_d),$$
$$(\beta_{ih}^{(d-1)} \alpha_j \mid e_1 e_2 \cdots e_d),$$

and

$$\left(\begin{array}{cc|cc} \beta_{i_1 h_1}^{(d-1)} & \beta_{i_d h_d} & e_1 & e_2 \cdots e_d \\ \beta_{i_2 h_2}^{(d-1)} & \beta_{i_d h_d} & e_1 & e_2 \cdots e_d \\ \cdots & \cdots & \cdots & \\ \beta_{i_{d-1} h_{d-1}}^{(d-1)} & \beta_{i_d h_d} & e_1 & e_2 \cdots e_d \end{array}\right).$$

Proof. We know that sl(V)-invariants of $\bigoplus_{\mathbf{n},\mathbf{k}} \mathrm{Super}^{[\mathbf{n}]}[\mathbb{L}_{\mathbf{k}} \mid \mathscr{P}]$ are linear combinations of the elements $(R \mid T)$, where T is a rectangular tableau with each row being $e_1^- e_2^- \cdots e_d^-$, and where $\mathrm{cont}(R, \beta_{ih})$ is either $d-1$ or 0 for each β_{ih}. It is sufficient to show that $(R \mid T)$ can be written as a linear combination of elements $(R' \mid T)$ such that each row of R' is either of the form $\alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_d}$, or of the form $\beta_{ih}^{(d-1)} \alpha_j$, or $\beta_{ih}^{(d-1)} \beta_{jk}$. Call such rows "good." Arrange the rows of R such that "good" rows are on the top. Then take the first row in R which is not "good" and let β_{ih} be a letter that appears s times in this row, where $1 \leqslant s < d-1$. Consider all the rows of $(R \mid T)$ containing β_{ih}; some of them may be "good" rows on the top. However, if a "good" row contains β_{ih}, then it is of the form $(\beta_{jk}^{(d-1)} \beta_{ih} \mid e_1 e_2 \cdots e_d)$. Collect all the rows of $(R \mid T)$ containing β_{ih},

$$\left(\begin{array}{cc|cc} \beta_{jk}^{(d-1)} & \beta_{ih} & e_1 & e_2 \cdots e_d \\ \cdots & & \cdots & \\ \beta_{j'k'}^{(d-1)} & \beta_{ih} & e_1 & e_2 \cdots e_d \\ \beta_{ih}^{(s)} & \omega & e_1 & e_2 \cdots e_d \\ \cdots & & \cdots & \\ \beta_{ih}^{(s')} & \omega' & e_1 & e_2 \cdots e_d \end{array}\right), \tag{55}$$

where the letter β_{ih} appears $d-1$ times, and where the top rows are "good".

Applying the exchange identity of the form

$$\left(\begin{array}{cc|ccc} \beta_{ih}^{(s)} & u & e_1 & e_2 \cdots e_d \\ \beta_{ih}^{(t)} & v & e_1 & e_2 \cdots e_d \end{array}\right) = \sum_u (-1)^{\mathrm{length}(u_{(2)})} \left(\begin{array}{cc|ccc} \beta_{ih}^{(s+t)} & u_{(1)} & e_1 & e_2 \cdots e_d \\ u_{(2)} & v & e_1 & e_2 \cdots e_d \end{array}\right),$$

we can collect all the $d-1$ occurrencies of β_{ih} to the first of the "non-good" row in (55) without splitting the words $\beta_{jk}^{(d-1)}, ..., \beta_{j'k'}^{(d-1)}$ in the top "good" rows; thus the tableau in (55) can be written as a linear combination of bitableaux of the form

$$\left(\begin{array}{c|c} \beta_{jk}^{(d-1)}u & e_1 \quad e_2 \cdots e_d \\ \cdots & \cdots \\ \beta_{j'k'}^{(d-1)}u' & e_1 \quad e_2 \cdots e_d \\ \beta_{ih}^{(d-1)}u'' & e_1 \quad e_2 \cdots e_d \\ v & e_1 \quad e_2 \cdots e_d \\ \cdots & \cdots \\ v' & e_1 \quad e_2 \cdots e_d \end{array}\right),$$

where $\mathrm{length}(u) = \cdots = \mathrm{length}(u') = \mathrm{length}(u'') = 1$, $\mathrm{length}(v) = \cdots = \mathrm{length}(v') = d$. Note that the number of "good" rows is increased at least by one, so the proposition follows by induction. ∎

EXAMPLE. Let $d=4$. We have

$$\left(\begin{array}{c|cccc} \beta_{11}^{(3)}\beta_{31} & e_1 & e_2 & e_3 & e_4 \\ \beta_{21}^{(3)}\beta_{31} & e_1 & e_2 & e_3 & e_4 \\ \beta_{31}\alpha_1^-\alpha_2^-\alpha_3^- & e_1 & e_2 & e_3 & e_4 \end{array}\right)$$

$$= \pm\sum_\omega \left(\begin{array}{c|cccc} \beta_{11}^{(3)}\beta_{31} & e_1 & e_2 & e_3 & e_4 \\ \beta_{21}^{(3)}\omega_{(2)} & e_1 & e_2 & e_3 & e_4 \\ \beta_{31}^{(2)}\omega_{(1)} & e_1 & e_2 & e_3 & e_4 \end{array}\right), \qquad \text{where } \omega = \alpha_1^-\alpha_2^-\alpha_3^-$$

$$= \pm\sum_\omega \sum_{\omega_{(1)}} \left(\begin{array}{c|cccc} \beta_{11}^{(3)}\omega_{(12)} & e_1 & e_2 & e_3 & e_4 \\ \beta_{21}^{(3)}\omega_{(2)} & e_1 & e_2 & e_3 & e_4 \\ \beta_{31}^{(3)}\omega_{(11)} & e_1 & e_2 & e_3 & e_4 \end{array}\right)$$

$$= \pm \sum_{\sigma\in S_3} (-1)^{|\sigma|} \left(\begin{array}{c|cccc} \beta_{11}^{(3)}\alpha_{\sigma(1)} & e_1 & e_2 & e_3 & e_4 \\ \beta_{21}^{(3)}\alpha_{\sigma(2)} & e_1 & e_2 & e_3 & e_4 \\ \beta_{31}^{(3)}\alpha_{\sigma(3)} & e_1 & e_2 & e_3 & e_4 \end{array}\right),$$

where the last sum consists of only "good" rows.

Since $\mathrm{Super}^{[\mathbf{n}]}[\mathbb{L}_{\mathbf{k}} \mid \mathscr{P}_y]$ is gl(V)-isomorphic to $\mathrm{Super}^{[\mathbf{n}]}[\mathbb{L}_{\mathbf{k}} \mid \mathscr{P}]$, the sl($V$)-invariants of the first are the sl(V)-invariants of the second multiplied by the factor $(\beta_{11} \mid y) \cdots (\beta_{1k_2} \mid y) \cdots (\beta_{n_21} \mid y) \cdots (\beta_{n_2k_2} \mid y)$. Furthermore, since the sl(V)-invariants of $\mathrm{Super}^{[\mathbf{n}]}[\mathscr{L}_{\mathbf{k}} \mid \mathbb{P}]$ are images of the sl(V)-invariants of $\mathrm{Super}^{[\mathbf{n}]}[\mathbb{L}_{\mathbf{k}} \mid \mathscr{P}_y]$ under $\mathfrak{U}_2$, we have

PROPOSITION 36. *Every* sl(V)*-invariant of* $\bigoplus_{\mathbf{n},\mathbf{k}} \mathrm{Super}^{[\mathbf{n}]}[\mathscr{L}_{\mathbf{k}} \mid \mathbb{P}]$ *is a polynomial in elements of the forms*

$$(\alpha_{i_1}\alpha_{i_2} \cdots \alpha_{i_d} \mid e_1 e_2 \cdots e_d),$$

$$(\beta^*_{i_1}\beta^*_{i_2} \cdots \beta^*_{i_d} \mid f_1 f_2 \cdots f_d),$$

$$(\alpha_i \mid \beta^*_j) \overset{\mathrm{def}}{=} \sum_s (\beta^*_j \mid f_s)(\alpha_i \mid e_s).$$

Proof. Consider the linear operator

$$\mathfrak{U}_2 : \bigoplus_{\mathbf{n},\mathbf{k}} \mathrm{Super}^{[\mathbf{n}]}[\mathbb{L}_{\mathbf{k}} \mid \mathscr{P}_y] \to \bigoplus_{\mathbf{n},\mathbf{k}} \mathrm{Super}^{[\mathbf{n}]}[\mathscr{L}_{\mathbf{k}} \mid \mathbb{P}].$$

It is sufficient to show that

$$\mathfrak{U}_2 \cdot (\alpha_{i_1}\alpha_{i_2} \cdots \alpha_{i_d} \mid e_1 e_2 \cdots e_d) = (\alpha_{i_1}\alpha_{i_2} \cdots \alpha_{i_d} \mid e_1 e_2 \cdots e_d), \tag{56}$$

$$\mathfrak{U}_2 \cdot (\beta_{ih} \mid y)(\beta_{ih}^{(d-1)}\alpha_j \mid e_1 e_2 \cdots e_d) = c \sum_s (\beta^*_i \mid f_s)(\alpha_i \mid e_s), \text{ for some } c \in \mathbb{K}, \tag{57}$$

$$\mathfrak{U}_2 \left((\beta_{i_1h_1} \mid y) \cdots (\beta_{i_dh_d} \mid y) \left(\begin{array}{ll|l} \beta_{i_1h_1}^{(d-1)} & \beta_{i_dh_d} & e_1 \quad e_2 \cdots e_d \\ \beta_{i_2h_2}^{(d-1)} & \beta_{i_dh_d} & e_1 \quad e_2 \cdots e_d \\ \cdots & \cdots & \cdots \\ \beta_{i_{d-1}h_{d-1}}^{(d-1)} & \beta_{i_dh_d} & e_1 \quad e_2 \cdots e_d \end{array} \right) \right)$$
$$= c \cdot (\beta^*_{i_1}\beta^*_{i_2} \cdots \beta^*_{i_d} \mid f_1 f_2 \cdots f_d), \quad \text{for some } c \in \mathbb{K}. \tag{58}$$

Formula (56) is immediate. To check (57), note that

$$\begin{aligned} &(\beta_{ih} \mid y)(\beta_{ih}^{(d-1)}\alpha_j \mid e_1 e_2 \cdots e_d) \\ &\quad = (\beta_{ih} \mid y)(-1)^{d-1} \sum_s (-1)^{d-s} (\beta_{ih}^{(d-1)} \mid e_1 \cdots \hat{e}_s \cdots e_d)(\alpha_j \mid e_s) \\ &\quad = \sum_s (-1)^{s-1} (\beta_{ih}^{(d)} \mid y e_1 \cdots \hat{e}_s \cdots e_d)(\alpha_j \mid e_s) \\ &\quad \xrightarrow{\mathfrak{U}_2} (-1)^{d-1} \sum (\beta^*_i \mid f_s)(\alpha_j \mid e_s). \end{aligned}$$

To see (58), note that the transformand on the left side equals

$$
\begin{aligned}
&(\beta_{i_1h_1} \mid y)\cdots(\beta_{i_dh_d} \mid y)\sum_{s_1}(-1)^{d-s_1}(\beta^{(d-1)}_{i_1h_1} \mid e_1\cdots\hat{e}_{s_1}\cdots e_d)(\beta_{i_dh_d} \mid e_{s_1})\\
&\qquad\cdots\sum_{s_{d-1}}(-1)^{d-s_{d-1}}(\beta^{(d-1)}_{i_{d-1}h_{d-1}} \mid e_1\cdots\hat{e}_{s_{d-1}}\cdots e_d)(\beta_{i_dh_d} \mid e_{s_{d-1}})\\
&\quad=\pm\sum_{s_i\neq s_j}(-1)^{d-s_1}(\beta^{(d)}_{i_1h_1} \mid ye_1\cdots\hat{e}_{s_1}\cdots e_d)\cdots(-1)^{d-s_{d-1}}\\
&\qquad\times(\beta^{(d)}_{i_{d-1}h_{d-1}} \mid ye_1\cdots\hat{e}_{s_{d-1}}\cdots e_d)(\beta^{(d)}_{i_dh_d} \mid ye_{s_1}\cdots e_{s_{d-1}})\\
&\quad=\pm\sum_{\sigma}(-1)^{|\sigma|}(-1)^{d-\sigma(1)}(\beta^{(d-1)}_{i_1h_1} \mid e_1\cdots\hat{e}_{\sigma(1)}\cdots e_d)\\
&\qquad\cdots(-1)^{d-\sigma(d-1)}(\beta^{(d-1)}_{i_{d-1}h_{d-1}} \mid e_1\cdots\hat{e}_{\sigma(d-1)}\cdots e_d)\\
&\qquad\times(-1)^{d-\sigma(d)}(\beta^{(d)}_{i_dh_d} \mid ye_1\cdots\hat{e}_{\sigma(d)}\cdots e_d);
\end{aligned}
\tag{59}
$$

the last step above follows from the identity

$$e_{\sigma(1)}e_{\sigma(2)}\cdots e_{\sigma(d-1)}=(-1)^{|\sigma|}(-1)^{d-\sigma(d)}e_1\cdots\hat{e}_{\sigma(d)}\cdots e_d.$$

The image of the expression in (59) under $\mathfrak{U}_2$ is

$$
\begin{aligned}
&\pm\sum_{\sigma}(-1)^{|\sigma|}(\beta^*_{i_1} \mid f_{\sigma(1)})\cdots(\beta^*_{i_{d-1}} \mid f_{\sigma(d-1)})(\beta^*_{i_d} \mid f_{\sigma(d)})\\
&\qquad=\pm(\beta^*_{i_1}\beta^*_{i_2}\cdots\beta^*_{i_d} \mid f_1f_2\cdots f_d). \quad\blacksquare
\end{aligned}
$$

Summarizing, we have obtained

THEOREM 13 (The First Fundamental Theorem). *The* $\mathrm{sl}(V)$*-invariants of* $\mathrm{Sym}^{n_1}(\mathrm{Sym}^{k_1}(V))\otimes\mathrm{Sym}^{n_2}(\Lambda^{k_2}(V)^*)$ *are of the form*

$$\mathfrak{U}_1((\alpha_1 \mid x_1)\cdots(\alpha_{n_1} \mid x_1)(\beta^*_1 \mid x_2)\cdots(\beta^*_{n_2} \mid x_2)\,\mathbf{m}),$$

where $\mathbf{m}\in\mathrm{Super}^{[\mathbf{n}]}[\mathscr{L}_{\mathbf{k}} \mid \mathbb{P}]$ *is a polynomial in elements of the forms*

$$
\begin{gathered}
(\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_d} \mid e_1e_2\cdots e_d),\\
(\beta^*_{i_1}\beta^*_{i_2}\cdots\beta^*_{i_d} \mid f_1f_2\cdots f_d),\\
(\alpha_i \mid \beta^*_j).
\end{gathered}
$$

At the end, we treat briefly the problem of the symbolic–umbral operator for general $\mathrm{Sym}(\bigoplus_i W_i)$, where each W_i is either $\mathrm{Sym}^{h_1}(V)$, or $\Lambda^{h_2}(V)$, or $\mathrm{Sym}^{k_1}(V)^*$, or $\Lambda^{k_2}(V)^*$. The dual bases $\mathscr{P}=\{e_1^-, e_2^-, ..., e_d^-\}$ and $\mathscr{P}^*=\{f_1^-, f_2^-, ..., f_d^-\}$ still remain negative. To define the first level symbolic–umbral operator $\mathfrak{U}_1$, we use negative letters for $\mathrm{Sym}^k(V)$ and

$\mathrm{Sym}^k(V)^*$ and positive letters for $\Lambda^k(V)$ and $\Lambda^k(V)^*$, as well as a mock place x_i^+ for each $\mathrm{Sym}^k(V)$ or $\mathrm{Sym}^k(V)^*$ and a mock place x_j (which is crucial when k is odd) for each $\Lambda^k(V)$ or $\Lambda^k(V)^*$ where $|x_j| = (-1)^k$. To define the second level symbolic–umbral operators $\mathfrak{U}_2$, we still associate to each letter $\beta_i^{*\pm}$ belonging to $\mathrm{Sym}^k(V)^*$ or $\Lambda^k(V)^*$ a set of letters $\{\beta_{i1}^+, \beta_{i2}^+, ..., \beta_{ik}^+\}$. For each component $\mathrm{Sym}^k(V)^*$, use a mock place y_i of the signature $|y_i| = (-1)^{d-1}$; and for each component $\Lambda^k(V)^*$ use a mock place y_i of the signature $|y_i| = (-1)^d$. Then the regularization algorithm and the first fundamental theorem for $\mathrm{Sym}(\bigoplus_i W_i)$ can be stated as before.

EXAMPLE (Joint Covariants [23, 36, 43]). We shall not discuss the role of mock places, which should be obvious in the present context. The case of classical covariants is that of the invariants of a plethystic algebra of the form $A = \mathrm{Sym}(\mathrm{Sym}^{k_1}(V)^* \oplus \cdots \oplus \mathrm{Sym}^{k_p}(V)^* \oplus \Lambda^{h_1}(V)^* \oplus \cdots \oplus \Lambda^{h_q}(V)^* \oplus V)$. The algebra A is the epimorphic image, under the first level operator $\mathfrak{U}_1$ of a subspace $\mathscr{A}$ of the (first level) symbolic letterplace algebra. The symbolic alphabet is made of letters β_s^* associated with the components $\mathrm{Sym}^{k_1}(V)^*, ..., \mathrm{Sym}^{k_p}(V)^*, \Lambda^{h_1}(V)^*, ..., \Lambda^{h_q}(V)^*$, as well as a letter α that represents the covariant component V. In the symbolic preimage $\mathscr{A}$, the letter α appears in a letterplace variable only when paired with a covariant place e_i, $i = 1, 2, ..., d$; on the other hand, the letters β_s^* associated with the contravariant components $\mathrm{Sym}^{k_1}(V)^*, ..., \Lambda^{h_q}(V)^*$ must be paired with contravariant places f_j, $j = 1, 2, ..., d$. Thus, in this special case, Proposition 36 implies that the joint covariants are evaluations under $\mathfrak{U}_1$ of polynomials in contragradient brackets $(\beta_{s_1}^* \beta_{s_2}^* \cdots \beta_{s_d}^* \mid f_1 f_2 \cdots f_d)$ and inner products $(\alpha \mid \beta_s^*)$.

APPENDIX. LEFT AND RIGHT SUPERDERIVATIONS ON SUPERSYMMETRIC ALGEBRAS

In this paper, we have used right superderivations. Left superderivation could have been used instead. We explain here formally that right superderivations and left superderivations are "equivalent."

Let $\mathscr{X} = \{x_1, x_2, ..., x_p\}$ be a $\mathbb{Z}_2$-graded alphabet. Consider the supersymmetric algebra $\mathrm{Super}[\mathscr{X}]$. For every $\omega = x_{i_1} x_{i_2} \cdots x_{i_{q-1}} x_{i_q}$, set

$$\varphi(\omega) = x_{i_q} x_{i_{q-1}} \cdots x_{i_2} x_{i_1}.$$

Clearly, the map φ extends linearly to an *even involution*

$$\varphi\colon \mathrm{Super}[\mathscr{X}] \to \mathrm{Super}[\mathscr{X}],$$

such that

$$\varphi(\omega_1 \omega_2) = \varphi(\omega_2)\, \varphi(\omega_1).$$

PROPOSITION 37. *Let* D: $\mathrm{Super}[\mathscr{X}] \to \mathrm{Super}[\mathscr{X}]$ *be a left superderivation. Then* $\varphi D \varphi$ *is a right superderivation (and vice versa).*

Proof.

$$\begin{aligned}(\varphi D\varphi)(\omega_1\omega_2) &= (\varphi D)(\varphi(\omega_2)\cdot\varphi(\omega_1))\\ &= \varphi(D\varphi(\omega_2)\cdot\varphi(\omega_1) + (-1)^{|D|\,|\omega_2|}\,\varphi(\omega_2)\cdot D\varphi(\omega_1))\\ &= \varphi^2(\omega_1)\cdot\varphi D\varphi(\omega_2) + (-1)^{|D|\,|\omega_2|}\,\varphi D\varphi(\omega_1)\cdot\varphi^2(\omega_2)\\ &= \omega_1\cdot\varphi D\varphi(\omega_2) + (-1)^{|D|\,|\omega_2|}\,\varphi D\varphi(\omega_1)\cdot\omega_2. \quad \blacksquare\end{aligned}$$

COROLLARY 38. *Let* $\mathfrak{L}$ *be a Lie superalgebra. Suppose* $\mathfrak{L}\bullet\mathrm{Super}[\mathscr{X}]$ *and* $\mathfrak{L}\circ\mathrm{Super}[\mathscr{X}]$ *are* $\mathfrak{L}$*-modules such that*

(a) *the action of* $\mathfrak{L}$ *on* $\mathfrak{L}\bullet\mathrm{Super}[\mathscr{X}]$ *is by left superderivation;*

(b) *the action of* $\mathfrak{L}$ *on* $\mathfrak{L}\circ\mathrm{Super}[\mathscr{X}]$ *is by right superderivation;*

(c) $D\bullet x_i = D\circ x_i$ *for each* $x_i\in\mathscr{X}$, $D\in\mathfrak{L}$.

Then $\mathfrak{L}\bullet\mathrm{Super}[\mathscr{X}]$ *and* $\mathfrak{L}\circ\mathrm{Super}[\mathscr{X}]$ *are* $\mathfrak{L}$*-isomorphic; i.e., they are equivalent representations of* $\mathfrak{L}$.

Proof. Let $D\in\mathfrak{L}$. We know that if $D\bullet$ acts as a left superderivation, then $\varphi D\varphi$ acts as a right superderivation. Since

$$(\varphi D\varphi)\,x_i = D\bullet x_i = D\circ x_i, \qquad \text{for all} \quad x_i\in\mathscr{X}$$

we have

$$(\varphi D\varphi)(\omega) = D\circ\omega, \qquad \text{for all} \quad \omega\in\mathrm{Super}[\mathscr{X}].$$

Therefore

$$D\bullet(\varphi\omega) = \varphi(D\circ\omega). \quad \blacksquare$$

According to this corollary, although we developed the theory where the $\mathrm{pl}(V)$-action is implemented by right superderivations, one can present a parallel theory using left superderivations. The two theories are *equivalent*, from the point of view of representation theory.

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